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SOME INEQUALITIES FOR SUMMABILITY METHODS IN WEIGHTED ORLICZ SPACES

Jafarov Sadulla[1,2]

[1]Department of Mathematics and Science Education, Faculty of Education, Muş Alparslan University, Turkey
[2]Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan , Baku, Azerbaijan
e-mail: s.jafarov@alparslan.edu.tr

Abstract. In this work the approximation of the functions by linear means of Fourier series in weighted Orlicz spaces was investigated. This result was applied to the approximation of the functions by linear means of Faber series in weighted Smirnov-Orlicz classes defined on simply connected domain of the complex plane.

Keywords: Young function, Boyd indices, Muckenhoupt weight, weighted Orlicz spaces, modulus of smoothness, Zygmund means, best approximation.

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1. Intrduction and main results

A convex and continuous function $M:[0,\infty)\to[0,\infty)$ for which $M(0)=0,\ \ M(x)>0$ for $x>0$ i

$$\lim_{x\to 0}\frac{M(x)}{x}=0, \qquad \lim_{x\to\infty}\frac{M(x)}{x}=\infty$$

İs called a *Young function*. The *complementary Young function* N of M is defined by

$$N(y):=\max_{x\ge 0}\left(xy-M(x)\right),\ \ y\ge 0.$$

Let T denote the interval $[-\pi,\pi]$, $\mathbb{C}$ the complex plane , and $L_p(T)$, $1\le p\le\infty$, the Lebesgue space of measurable complex-valued functions on T.

For a given Young function M let $\tilde{L}_M(T)$ denote the set of all Lebesgue measurable functions $f:T\to\mathbb{C}$ for which

$$\int_T M\left[\left|f(x)\right|\right]dx<\infty .$$

Let N be the complementary Young function of M. It is well-known [26, p. 69], [39, pp. 52-68] that the linear span of $\tilde{L}_M(T)$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(T)} := \sup\left\{\int_T |f(x)g(x)|dx : g \in \tilde{L}_N(T),\ \int_T N(|g(x)|)dx \le 1\right\},$$

or with the *Luxemburg norm*

$$\|f\|^*_{L_M(T)} := \inf\left\{k>0 : \int_T M\left(\frac{|f(x)|}{k}\right)dx \le 1\right\}$$

becomes a Banach space. The space is denoted by $L_M(T)$ and is called an *Orlicz space* [24, p.26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(T),\ 1<p<\infty$. If $M(x) := M(x,p) = x^p,\ 1<p<\infty,$ then Orlicz spaces $L_M(T)$ coindices with the usual Lebesgue spaces $L_p(T),\ 1<p<\infty$. Note that the Orlicz spaces play an important role in many areas such as applied mathematics, mechanics, regularirty theory, fluid dynamics and statistical physics. Therefore, the approximation of the functions by means of Fourier trigonıometric series in Orlicz spaceas is also important in these areas of research.

The Luxemburg norm is equivalent to the Orlicz norm and equivalence

$$\|f\|^*_{L_M(T)} \le \|f\|_{L_M(T)} \le 2\|f\|^*_{L_M(T)},\quad f \in L_M(T)$$

holds true [24, p.80].

If we choose $M(u) = \frac{u^p}{p}$ $,1<p<\infty$ then the complementary function is $N(u) = \frac{u^q}{q}$ with $\frac{1}{p}+\frac{1}{q}=1$ and we have the relation

$$p^{-1/p}\|u\|_{L_p(T)} = \|u\|^*_{L_M(T)} \le \|u\|_{L_M(T)} \le q^{1/q}\|u\|_{L_p(T)},$$

where $\|u\|_{L_p(T)} = \left(\int_T |u(x)|^p dx\right)^{1/p}$ stands for the usual norm of the $L_p(T)$ space.

If N is complementary to M in Young's sense and $f \in L_M(T),\ g \in L_N(T)$ then the so-called strong Hölder inequalities [26, p. 80]

$$\int_T |f(x)g(x)|dx \le \|f\|_{L_M(T)}\|g\|^*_{L_N(T)},$$

$$\int_T |f(x)g(x)|dx \le \|f\|^*_{L_M(T)}\|g\|_{L_N(T)},$$

are satisfied.
An N function M satisfies the Δ_2-*condition* if

$$\limsup_{x\to\infty}\frac{M(2x)}{M(x)}<\infty\ .$$

The Orlicz space $L_M(T)$ is *reflexive* if and only if the N - function M and its complementary function N both satisfy the Δ_2-condition [39, p.113]. Furter information about Orlicz spaces may be found in [26] and [39].

Let $M^{-1}:[0,\infty)\to[0,\infty)$ be the inverse function of the N function M. The *lower* and *upper indices* α_M, β_M [7, p.350]

$$\alpha_M:=\lim_{x\to 0}\frac{\log h(x)}{\log x},\quad \beta_M:=\lim_{x\to\infty}\frac{\log h(x)}{\log x}$$

of the function

$$h:(0,\infty)\to(0,\infty],\qquad h(x):=\limsup_{t\to\infty}\frac{M^{-1}(t)}{M^{-1}\left(\frac{t}{x}\right)},\ \ x>0$$

firs considered by W. Matuszewska ve W. Orlicz [32], are called the *Boyd indices* of the Orlicz space $L_M(T)$. It is known that

$$0\le\alpha_M\le\beta_M\le 1$$

and

$$\alpha_N+\beta_M=1,\qquad\qquad \alpha_M+\beta_N=1.$$

The Boyd indices α_M, β_M are called *nontrivial* if $0<\alpha_M$ and $\beta_M<1$. The Orlicz space $L_M(T)$ is reflexive if and only if $0<\alpha_M\le\beta_M<1$, i.e. if the Boyd indices are nontrivial. The detailed information about Boyd indices can be found in [4-10], [27] and [36].

A measurable function $\omega:\ T\to[0,\infty]$ is called *a weight function* if the set $\omega^{-1}(\{0,\infty\})$ has Lebesgue measure zero. With any given weight ω we associate the ω-*weight Orlicz space* $L_M(T,\omega)$ consisting of all measurable function f on T such that

$$\|f\|_{L_M(T,\omega)}:=\|f\omega\|_{L_M(T)}\ .$$

Note that if $\omega\in L_M(T)$ and $1/\omega\in L_N(T)$ then from Hölder inequality we have $L_\infty(T)\subset L_M(T,\omega)\subset L_1(T)$.

Let $1 < p < \infty$, $\frac{1}{p}+\frac{1}{p'}$ and let ω be a weight function on T . ω is said to satisfy Muckenhoupt's A_p - condition on T if

$$\sup_J \left(\frac{1}{|J|} \int_J \omega^p(t)\,dt \right)^{1/p} \left(\frac{1}{|J|} \int_J \omega^{-p'}(t)\,dt \right)^{1/p'} < \infty,$$

where J is any subinterval of T and $|J|$ denotes its length [33].

Let us indicate by $A_p(T)$ the set of all weight functions satisfying Muckenhoupt's A_p - condition on T .

According to [28, Lemma 3.3], [27, Theorem 4.5], [30, Section 2.5], if $L_M(T)$ is reflexive and ω weight function satisfying the condition $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$, then the space $L_M(T,\omega)$ is also reflexive.

Let $L_M(T,\omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \le \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. For a given $f \in L_M(T,\omega)$, the shift operatör s_h is defined by $(s_h f)(x) := \frac{1}{2h}\int_{-h}^{h} f(x+t)\,dt,\ 0 < h < \pi,\ x \in T$.

We define k *-modulus of smoothness* for $f \in L_M(T,\omega)$ as

$$\Omega^k_{M,\omega}(\delta, f) := \sup_{\substack{0<h_i\le\delta \\ 1\le i\le k}} \left\| \prod_{i=1}^{k} \left(I - s_{h_i} \right) \right\|_{L_M(T,\omega)},\ \ \delta > 0,$$

where I is the identity operatör. Note that this modulus of smoothness is well defined, because s_h is a bounded linear operatör on $L_M(T,\omega)$ [20].

It is easily verified that the function $\Omega^k_{M,\omega}(\cdot, f)$ is continuous, non-negative and satisfy

$$\lim_{\delta\to 0} \Omega^k_{M,\omega}(\cdot, f) = 0,\quad \Omega^k_{M,\omega}(\cdot, f+g) \le \Omega^k_{M,\omega}(\cdot, f) + \Omega^k_{M,\omega}(\cdot, g)$$

for $f, g \in L_M(T,\omega)$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k(f)\cos kx + b_k(f)\sin kx \right) \tag{1.1}$$

be the Fourier series of the function $f \in L_1\left(T,\omega\right)$, where $a_k\left(f\right)$ and $b_k\left(f\right)$ are the Fourier coefficiets of the function of f. For $f \in L_M\left(T,\omega\right)$ we define the summability method by the tringular matrix $\Lambda = \left\{\lambda_{i,j}\right\}_{i,j=0}^{j,\infty}$ by the linear means ,

$$U_n\left(x,f\right) = \lambda_{0n}\frac{a_0}{2} + \sum_{i=1}^{n}\lambda_i\left(a_{i,}\left(f\right)\cos ix + b_i\left(f\right)\sin ix\right).$$

If the Fourier series of is given by (1.1). then Zygmund-Riesz means of order k is defined as

$$Z_n^k\left(x,f\right) = \frac{a_0}{2} + \sum_{i=1}^{n}\left(1 - \frac{i^k}{\left(n+1\right)^k}\right)\left(a_{i,}\left(f\right)\cos ix + b_i\left(f\right)\sin ix\right).$$

We denote by $E_n\left(f\right)_{M,\omega}$, $\left(n = 0,1,2,...\right)$ the best approximation of $f \in L\left(T,\omega\right)$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n\left(f\right)_{M,\omega} := \inf\left\{\left\|f - T_n\right\|_{L_M\left(T,\omega\right)} : T_n \in \Pi_n\right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n. Note that the existence of $T_n^* \in \Pi_n$ such that

$$E_n\left(f\right)_{M,\omega} = \left\|f - T_n^*\right\|_{L_M\left(T,\omega\right)}$$

follows, for example, from Theorem 1.1 in [14, p.59].
Let $T_n \in \Pi_n$ and

$$T_n = \frac{c_0}{2} + \sum_{i=1}^{n}\left(c_i\cos ix + d_i\sin ix\right).$$

Then conjugate polynomial $\tilde{T}_n$ is defined by

$$\tilde{T}_n = \sum_{i=1}^{n}\left(c_i\sin ix - d_i\cos ix\right).$$

In this paper we use the constants $c, c_1, c_2,...$(in general, different in different relations) which depend only on the quantities that are not important fort the questiones of interest.

We will say that the method of summability by the matrix condition Λ satisfies condition $b_{k,M}$ (respectively $b_{k,M}^*$) if for $T_n \in \Pi_n$ the inequality

$$\left\|T_n - U_n\left(T_n\right)\right\|_{L_M(T,\omega)} \le c\left(n+1\right)^{-k}\left\|T_n^{(k)}\right\|_{L_M(T,\omega)}$$

$$\left(\left\|T_n - U_n\left(T_n\right)\right\|_{L_M(T,\omega)} \le c\left(n+1\right)^{-k}\left\|\tilde{T}_n^{(k)}\right\|_{L_M(T,\omega)}\right)$$

holds and the norms

$$\left\|\Lambda\right\|_1 := \int_0^{2\pi}\left|\frac{\lambda_{0,n}}{2} + \sum_{i=1}^{n}\lambda_{i,n}\cos it\right|dt$$

are bounded .

The problems problems of approximation theory in the weighted and non-weighted Orlicz spaces have been investigated by several authors (see, for example. [1-3 , 15, 18, 20, 22, 24, 38].

In the present paper necessary ant sufficient condition abouıt the relationship between the approximation of the functions by linear means of Fourier series and by Zygmund-Riesz means of order k was investigated in weighted Orlicz spaces. Also, we investigate the approximation of functions by linear means of Fourier series in terms of the modulus of smoothness of these functions in weighted Orlicz spaces. This result was applied to the approximation of the functions by linear means of Faber series in weifhted Smirnov-Orlicz classes defined on simply connected domain of the complex plane. The similar problems in different spaces were investigated in [11, 12, 21, 23, 25, 32, 33, 41, 42]

Main results in the present work are the following theorems:

Theorem 1.1. Let $L_M\left(T,\omega\right)$ be a weighed Orlicz spaces with Boyd indices $1<\alpha_M \le \beta_M <1$, and let $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$. In order that for $f \in L_M\left(T,\omega\right)$

$$\left\|f\left(\cdot\right) - U_n\left(\cdot,f\right)\right\|_{L_M(T,\omega)} \le c_1\left\|f\left(\cdot\right) - Z_n^k\left(\cdot,f\right)\right\|_{L_M(T,\omega)} \tag{1.2}$$

it is sufficient and necessary that for $f \in L_M\left(T,\omega\right)$

$$\left\|T_n\left(\cdot\right) - U_n\left(\cdot,f\right)\right\|_{L_M(T,\omega)} \le c_2\left\|T_n\left(\cdot\right) - Z_n^k\left(\cdot,T_n\right)\right\|_{L_M(T,\omega)}. \tag{1.3}$$

Theorem 1.2. Let $L_M\left(T,\omega\right)$ be a weighed Orlicz spaces with Boyd indices $1<\alpha_M \le \beta_M <1$, and let $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$. In order that for every $f \in L_M\left(T,\omega\right)$

$$\left\|f\left(\cdot\right) - U_n\left(\cdot,f\right)\right\|_{L_M(T,\omega)} \le c_3\left\|f\left(\cdot\right) - Z_n^k\left(\cdot,f\right)\right\|_{L_M(T,\omega)} \tag{1.4}$$

it is sufficient and necessary that

(i) $\left\|U_n\left(\cdot,f\right)\right\|_{L_M(T,\omega)} = O(1)$;

(ii) if k is even, $U_n(\cdot, f)$ satisfies the condition $(b_{k,M})$; if k is odd $U_n(\cdot, f)$ satisfies the condition $(b_{k,M}^*)$.

Theorem 1.3. Let $L_M(T,\omega)$ be a weighed Orlicz spaces with Boyd indices $1<\alpha_M \le \beta_M <1$, and let $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$. If the summability method with the matrix Λ satisfies the condition $(b_{k,M})$ or $b_{k,M}^*$, then for $f \in L_M(T,\omega)$ the inequality

$$\left\| f(\cdot)-U_n(\cdot, f)\right\|_{L_M(T,\omega)} \le c_4 \Omega_{M,\omega}^k\left(\frac{1}{n+1}, f\right) \qquad (1.5)$$

holds with a constant $c_4>0$ independent of n.

Theorem 1.4. Let $L_M(T,\omega)$ be a weighed Orlicz spaces with Boyd indices $1<\alpha_M \le \beta_M <1$, and let $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$. If the summability method with the matrix Λ satisfies the condition $(b_{k,M})$ or $(b_{k,M}^*)$, then for $f \in L_M(T,\omega)$

$$\Omega_{M,\omega}^k(\delta, U_n(\cdot, f)) \le c_5 \Omega_{M,\omega}^k(\delta, f) \qquad (1:6)$$

where the constant $c_5>0$ does not depend on n, f and δ.

Corollary 1.1. These results obtained in Theorems 1.1 and 1.3 are valid Zygmund- Riesz means of order k.

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ, and let $G^- := ext\Gamma$. Further let

$$T. = \{w \in \mathbb{C} : |w| = 1\},\ D := \operatorname{int} T \text{ and } D^- := extT.$$

Let $w=\phi(z)$ be the conformal mapping of G^- onto D^- normalized by

$$\phi(\infty)=\infty,\ \lim_{z\to\infty}\frac{\phi(z)}{z}>0,$$

and let ψ denote the inverse of ϕ.

Let $\omega=\phi_1(z)$ denote a function that maps the domain G conformaly onto the disk $|w|<1$. The inverse mapping of ϕ_1 wil be denoted by ψ_1. Let Γ_r denote circular images in the domain G, that is, curves in G corresponding to circle $|\phi_1(z)|=r$ under the mapping $z=\psi_1(w)$.

Let us denote by $E^p(G)$, where $p>0$, the class of al functions $f(z)\neq 0$ which are analytic in G and have the property that the integral

$$\int_{\Gamma_r} |f(z)|^p |dz|$$

İs bounded for $0<r<1$. We shall call the $E^p(G)$-class the *Smirnov class*. Ever function in $E^p(G), 1\leq p<\infty$, has the non-tangential boundary valurs almost everywhere (a.e.) on Γ and the boundary function belongs to Lebesgue space $L^p(\Gamma)$ [16, p.438]. It is known that $\phi' \in E^1(G^-)$ and $\psi' \in E^1(D^-)$. Note that the general information about Smirnov classes can be found in the books [13, pp.168-185] and [16, pp.438-453].

We define also the ω-weighted Smirnow-Orlicz class $E_M(G,\omega)$ as

$$E_M(G,\omega):=\left\{ f\in E^1(G): f\in L_M(\Gamma,\omega)\right\}.$$

Note that the weighted Smirnov-Orlicz class $E_M(G,\omega)$ is a generalization of the Smirnov class $E^p(G)$. In particular, If $M(x)=x^p$, $1<p<\infty$, then the weighted Smirnov-Orlicz class $E_M(G,\omega)$ coincides with the weighted Smirnov class $E^p(G,\omega)$; if $\omega=1$, then $E_M(G,\omega)$ coincides with the Smirnov-Orlicz class $E_M(G)$, defined in [31].

With every weight function ω on Γ, we associate another weight ω_0 on T defined by

$$\omega_0(t):=\omega(\psi(t)),\ t\in T.$$

For $f\in L_M(\Gamma,\omega)$ we define the function

$$f_0(t):=f(\psi(t)),\ t\in T.$$

Let h be continuous function on $[0,2\pi]$. Its modulus of continuity is defined by

$$\omega(t,h):=\sup\left\{ |h(t_1)-h(t_2)| : t_1,\ t_2\in[0,2\pi], |t_1-t_2|\leq t\right\},\ t\geq 0.$$

The curve Γ is called Dini-smooth if it has a parameterization

$$\Gamma:\varphi_0(s),\ 0\leq s\leq 2\pi$$

such that $\varphi_0'(s)$ is Dini-continuous,i.e.

$$\int_0^{\pi} \frac{\omega(t,\varphi_0')}{t} dt < \infty$$

and $\varphi_0'(s) \neq 0$ [37, p. 48].

If Γ is Dini -smooth curve, then there exist the constants c_6 and c_7 such that [41]

$$0 \leq c_6 \leq |\psi'(t)| \leq c_7 < \infty, \ |t| > 1 \ . \quad (1.7)$$

Note that If Γ is Dini -smooth curve, then by (1.7) we have $f_0 \in L_M(T,\omega_0)$ and $f \in L_M(\Gamma,\omega)$.

Let Γ be a rectifiable Jordan curve and $f \in L^1(\Gamma)$. Then the function f^+ defined by

$$f^+(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s-z} ds, \ z \in G$$

İs analytic in G. Note that İf $0 < \alpha_M, \ \beta_M < 1, \ \omega \in A_{\frac{1}{\alpha_M}}(\Gamma) \cap A_{\frac{1}{\beta_M}}(\Gamma)$ and $f \in L_M(\Gamma,\omega)$, then according to [16] $f^+ \in E_M(G,\omega)$.

Let $\phi_k(z), \ k = 0,1,2,\ldots$ be the Faber polynomials for G. The Faber polynomials $\phi_k(z)$, associated with $G \cup \Gamma$, are defined through the expansion

$$\frac{\psi'(t)}{\psi(t)-z} = \sum_{k=0}^{\infty} \frac{\phi_k(z)}{t^{k+1}}, \ z \in G, \ t \in D^- \quad (1.8)$$

and the equalities

$$\phi_k(z) = \frac{1}{2\pi i} \int_T \frac{t^k \psi'(t)}{\psi(t)-z} dt, \quad z \in G, \quad (1.9)$$

$$\phi_k(z) = \phi^k(z) + \frac{1}{2\pi i} \int_T \frac{\phi^k(s)}{s-z} dt, \quad z \in G^-$$

hold [40, pp. 33-48].

Let $f \in E_M(G,\omega)$. Since $f \in E^1(G)$ we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)ds}{s-z} = \frac{1}{2\pi i} \int_T \frac{f(\psi(t))\psi'(t)}{\psi(t)-z} dt$$

for every $z \in G$. Conssidering this formula and expansion (1.8) , we can associate with f the formal series

$$f(z) \sim \sum_{i=0}^{\infty} a_i(f)\phi_i(z), \ z \in G \qquad (1.10)$$

where

$$a_i(f) := \frac{1}{2\pi i}\int_T \frac{f(\psi(t))}{t^{i+1}}dt, \ i = 0,1,2,...$$

This series is called the *Faber series* expansion of f, and the coefficients $a_i(f)$ are said to be the *Faber coefficients* of f.

Let (1.10) be the Faber series of the function $f \in E_M(G,\omega)$. Fort he function f we define the summability method by the tringular matrix $\Lambda = \{\lambda_{i,j}\}_{i,j=0}^{j,\infty}$ by the linear means

$$U_n(z,f) = \sum_{i=0}^{n} \lambda_{i,n} a_i(f)\phi_i(z).$$

The n*-th partial sums* and *Zygmund means of order* k of the series (1.10) are defined respectively, as

$$S_n(z,f) = \sum_{k=0}^{n} a_k(f)\phi_k(z),$$

$$Z_n^k(z,f) = \sum_{i=0}^{n}\left(1 - \frac{i^k}{(n+1)^k}\right)a_k(f)\phi_i(z).$$

Let Γ be a Dini-smooth curve. Using the nontangential boandary values of f_0^+ on T we define the k*-th modulus of smoothness* of $f \in L_M(\Gamma,\omega)$ as $\Omega_{\Gamma,M,\omega}^k(f,\delta) := \Omega_{M,\omega_0}^k(f_0^+,\delta), \ \delta > 0$ for $k = 1,2,3,...$

The following theorem holds.

Theorem 1.5. Let Γ be a Dini-smoth curve, and let $L_M(\Gamma)$ be a reflexive Orlicz space, and let $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$. If the summability method with the matrix Λ satisfies the condition $(b_{k,M})$ or $(b_{k,M}^*)$, then for $f \in E_M(G,\omega)$ the estimate

$$\left\| f(\cdot) - U_n(\cdot,f) \right\|_{L_M(\Gamma,\omega)} \le c_8 \Omega_{\Gamma,M,\omega}^k\left(f, \frac{1}{n+1}\right) \qquad (1.11)$$

holds with a constant $c_8 > 0$, independent of n.

2. Some auxiliary results

Let $\wp$ be the set of all algebraic polynomials (with no restriction on the degree), and let $\wp(D)$ be the set of traces of members of $\wp$ on D. We define the operatör

$$A:\wp(D)\to E_M(G,\omega)$$

as

$$A(P)(z):=\frac{1}{2\pi i}\int_T \frac{P(w)\psi'(w)}{w-z}dw,\ \ z\in G .$$

Then it is clear that by (1.9) we get

$$A\left(\sum_{k=0}^{n}\beta_k w^k\right)=\sum_{k=0}^{n}\beta_k\phi_k(z).$$

The following results hold for the linear operator A [18].

Theorem 2.1. If Γ is a Dini-smoth curve, $0<\alpha_M,\ \beta_M<1,$ and $\omega\in A_{1/\alpha_M}(\Gamma)\cap A_{1/\beta_M}(\Gamma)$. Then linear operator $A:\wp(D)\to E_M(G,\omega)$ is bounded.

Theorem 2.2. If Γ is a Dini-smoth curve, $0<\alpha_M,\ \beta_M<1,$ and $\omega\in A_{1/\alpha_M}(\Gamma)\cap A_{1/\beta_M}(\Gamma)$. Then linear operator $A:E_M(D,\omega_0)\to E_M(G,\omega)$ is one-to-one and onto.

3. Proofs of the main results

Proof of Theorem 1.1. Necessity. It is clear that the inequality (1.3) follows from the inequality (1.2).

Sufficiency. Let $f\in L_M(\Gamma,\omega)$ and let $T_n\in\Pi_n$ $(n=0,1,2,...)$ be the polynomial of best approximation to f. We obtain

$$\begin{aligned}
&\left\|f-U_n(\cdot,f)\right\|_{L_M(T,\omega)}\\
&\le\left\|f-T_n\right\|_{L_M(T,\omega)}-\left\|T_n-U_n(\cdot,f)\right\|_{L_M(T,\omega)}+\left\|U_n(\cdot,f-T_n)\right\|_{L_M(T,\omega)}\\
&\le E_n(f)_{M,\omega}+c_9\left\|T_n-Z_n^k(\cdot,T_n)\right\|_{L_M(T,\omega)}+c_{10}E_n(f)_{M,\omega}\le c_{11}E_n(f)_{M,\omega}\\
&+c_{12}\left(\left\|T_n-f\right\|_{L_M(T,\omega)}+\left\|f-Z_n^k(\cdot,f)\right\|_{L_M(T,\omega)}+\left\|Z_n^k(\cdot,f-T_n)\right\|_{L_M(T,\omega)}\right)\\
&\le c_{13}E_n(f)_{M,\omega}+c_{14}E_n(f)_{M,\omega}+c_{15}\left\|f-Z_n^k(\cdot,f)\right\|_{L_M(T,\omega)}+c_{16}c_{17}E_n(f)_{M,\omega}\\
&\le c_{18}E_n(f)_{M,\omega}+c_{19}\left\|f-Z_n^k(\cdot,f)\right\|_{L_M(T,\omega)}\le c_{20}\left\|f-Z_n^k(\cdot,f)\right\|_{L_M(T,\omega)}
\end{aligned}$$

and Theorem 1.1 is proved.

Proof of Theorem 1.2. Necessity. Let $1<q<p<\infty$. According to [42] $Z_n^k(\cdot,f)$ is bounded in the Lebesgue spaces $L_p(T)$ and $L_q(T)$. Then by [20, Theorem 7 and Lemma 1] $\left\|Z_n^k(\cdot,f)\right\|_{L_M(T,\omega)}=O(1)$. Taking account of (1.4) we have $\left\|U_n(\cdot,f)\right\|_{L_M(T,\omega)}=O(1)$. Let $f\in L_M(T,\omega)$. Then the following inequality holds :

$$\begin{aligned}&\left\|f-Z_n^k(\cdot,f)\right\|_{L_M(T,\omega)}\\&\le\left\|f-S(\cdot,f)\right\|_{L_M(T,\omega)}+(n+1)^{-k}\left\|\sum_{\nu=1}^{n}\nu^kA_\nu(\cdot,f)\right\|_{L_M(T,\omega)}\qquad(2.1)\\&=U_1+U_2^{(k)}.\end{aligned}$$

It is will known from [20] that

$$\begin{aligned}U_1&=\left\|f-S(\cdot,f)\right\|_{L_M(T,\omega)}\le c_{21}E_n(f)_{M,\omega}\\&\le c_{22}(n+1)^{-k}\left\|f^{(k)}\right\|_{L_M(T,\omega)}.\end{aligned}\qquad(2.2)$$

We note that if k is even

$$\sum_{\nu=1}^{n}\nu^kA_\nu(x,f)=(-1)^{k/2}S_n^{(k)}(x,f),$$

if k is odd

$$\sum_{\nu=1}^{n}\nu^kA_\nu(x,f)=(-1)^{(k+3)/2}\tilde{S}_n^{(k)}(x,f),$$

where $\tilde{g}(x)$ is the function that is trigonometricaly conjugate to $g(x)$. Then

$$U_2^{(k)}=\begin{cases}(n+1)^{-k}\left\|S_n^{(k)}(\cdot,f)\right\|_{L_M(T,\omega)}, & k-even\\(n+1)^{-k}\left\|\tilde{S}_n^{(k)}(\cdot,f)\right\|_{L_M(T,\omega)}, & k-odd.\end{cases}\qquad(2.3)$$

Using (2.3) and [20], if k is even we have

$$\begin{aligned}U_2^{(k)}&=(n+1)^{-k}\left\|S_n^{(k)}(\cdot,f)\right\|_{L_k(T,\omega)}\\&\le c_{23}(n+1)^{-k}\left\|f^{(k)}\right\|_{L_M(T,\omega)},\end{aligned}\qquad(2.4)$$

if k is odd, we find that

$$U_2^{(k)} = (n+1)^{-k} \left\| \tilde{S}_n^{(k)}(\cdot, f) \right\|_{L_k(T,\omega)} \le c_{24} (n+1)^{-k} \left\| \tilde{f}^{(k)} \right\|_{L_M(T,\omega)} . \tag{2.5}$$

Taking into account the relations (2.1), (2.2), (2.2), and (2.5), if k is even for $f \in L_M(T,\omega)$ and $f^{(k)} \in L_M(T,\omega)$ we obtain the inequality

$$\left\| f - Z_n^{(k)}(\cdot, f) \right\|_{L_M(T,\omega)} \le c_{25} (n+1)^{-k} \left\| f^{(k)} \right\|$$

and if k is odd for $f \in L_M(T,\omega)$ and $\tilde{f}^{(k)} \in L_M(T,\omega)$ we reach

$$\left\| f - Z_n^{(k)}(\cdot, f) \right\|_{L_M(T,\omega)} \le c_{26} (n+1)^{-k} \left\| \tilde{f}^{(k)} \right\|.$$

Sufficienty. We note that for $T_n \in \Pi_n$ we get

$$\left\| T_n - Z_n^{(k)}(\cdot, T_n) \right\|_{L_M(T,\omega)} \le (n+1)^{-k} \left\| T_n^{(k)} \right\|_{L_M(T,\omega)}, \text{ if } k \text{ is even,} \tag{2.6}$$

$$\left\| T_n - Z_n^{(k)}(\cdot, T_n) \right\|_{L_M(T,\omega)} \le (n+1)^{-k} \left\| \tilde{T}_n^{(k)} \right\|_{L_M(T,\omega)}, \text{ if } k \text{ is odd.} \tag{2.7}$$

Use of (2.6), (2.7) and condition (ii) gives us

$$\left\| T_n - U_n(\cdot, T_n) \right\|_{L_M(T,\omega)} \le c_{27} \left\| T_n - Z_n^{(k)}(\cdot, T_n) \right\|_{L_M(T,\omega)} .$$

The last inequality and Theorem 1.1 imply that (1.4). Theorem 1.2 is completely proved.

Proof of Theorem 1.3. We suppose that the condition $(b_{k,M}^{*})$ is satisfied. Let $f \in L_M(T,\omega)$ and $T_n \in \Pi_n$ be the polynomial of best approximation to f. Note that $U_n(f) = \Lambda * f$. Then if $1 < q < p < \infty$ the operatör $U_n(f)$ is bounded in the Lebesgue spaces $L_p(T)$ and $L_q(T)$ [42]. Using the method of proof of Lemma 1 in [20] we can show that the operatör $U_n(f)$ is bounded in $L_M(T,\omega)$, i.e. $\left\| U_n(f) \right\|_{L_M(T,\omega)} \le c_{28} \| f \|_{L_M(T,\omega)}$. Then we get

$$\begin{aligned}
&\left\| f - U_n(\cdot, f) \right\|_{L_M(T,\omega)} \\
&\le \left\| f - T_n \right\|_{L_M(T,\omega)} - \left\| T_n - U_n(\cdot, f) \right\|_{L_M(T,\omega)} + \left\| U_n(\cdot, T_n) - U_n(\cdot, f) \right\|_{L_M(T,\omega)} \\
&\le c_{29} E_n(f)_{M,\omega} + c_{30} E_n(f)_{M,\omega} + c_{31} n^{-k} \left\| \tilde{T}_n^{(k)} \right\|_{L_M(T,\omega)} \\
&\le c_{32} E_n(f)_{M,\omega} + c_{33} n^{-k} \left\| \tilde{T}_n^{(k)} \right\|_{L_M(T,\omega)} .
\end{aligned} \tag{2.8}$$

Taking the Bernstein inequality and the boundedness of the linear operatör $f \to \tilde{f}$ in $L_M(T,\omega)$ into account [20, Lemma 3 and relation (15)] we have

$$\begin{aligned}
n^{-k} \left\| \tilde{T}_n^{(k)} \right\|_{L_M(T,\omega)} &\le c_{34} n^{-k} \left\| T_n^{(k)} \right\|_{L_M(T,\omega)} \\
&\le c_{35} \Omega_{M,\omega}^k \left(\frac{1}{n+1}, f \right).
\end{aligned} \tag{2.9}$$

Note that according to the direct theorem of approximation in $L_M(T,\omega)$ given in [20] following inequality holds:

$$E_n(f)_{M,\omega} \le c_{36} \Omega_{M,\omega}^k \left(\frac{1}{n+1}, f \right).$$

Taking into account the relations ( ), ( ) and ( ) we have

$$\left\| f - U_n(\cdot, f) \right\|_{L_M(T,\omega)} \le c_{37} \Omega_{M,\omega}^k \left(\frac{1}{n+1}, f \right). \tag{2.10}$$

If the summability method with the matrix Λ satisfies condition $\left(b_{k,M}^{*} \right)$, the proof is made anologously to the above. The proof of Theorem 1.3 is completed.

Proof of Theorem 1.4. By [20] the inequality

$$\Omega_{M,\omega}^k \left(\delta,\ U_n(f) - f, \right) \le c_{38} \left\| U_n(\cdot, f) - f \right\|_{L_M(T,\omega)} \tag{2.11}$$

holds.

Let $\delta \ge (n+1)^{-1}$. Using Theorem 1.3 and (2.11) we have

$$\Omega_{M,\omega}^{k}\left(\delta,U_{n}\left(f\right)\right)\leq\Omega_{M,\omega}^{k}\left(\delta,f\right)+\Omega_{M,\omega}^{k}\left(\delta,U_{n}\left(\cdot,f\right)-f\right)_{L_{M}\left(T,\omega\right)}$$
$$\leq\Omega_{M,\omega}^{k}\left(\delta,f\right)+c_{39}\left\|U_{n}\left(\cdot,f\right)-f\right\|_{L_{M}\left(T,\omega\right)}$$
$$\leq\Omega_{M,\omega}^{k}\left(\delta,f\right)+c_{40}\Omega_{M,\omega}^{k}\left(\frac{1}{n+1},f\right)\leq c_{41}\Omega_{M,\omega}^{k}\left(\delta,f\right). \qquad (2.12)$$

Now we suppose that $\delta<\left(n+1\right)^{-1}$. Then by virtue of [16] and [18] we obtain

$$\Omega_{M,\omega}^{k}\left(\delta,U_{n}\left(f\right)\right)\leq c_{42}\delta^{k}\left\|U_{n}^{(k)}\left(\cdot,f\right)\right\|_{L_{M}\left(T,\omega\right)}$$
$$\leq c_{43}\delta^{k}n^{k}\left\|U_{n}\left(\cdot,f\right)\right\|_{L_{M}\left(T,\omega\right)}\leq c_{44}\delta^{k}n^{k}\Omega_{M,\omega}^{k}\left(\frac{1}{n},U_{n}\left(f\right)\right)$$
$$\leq c_{45}\delta^{k}\left(n+1\right)^{k}\Omega_{M,\omega}^{k}\left(\frac{1}{n},U_{n}\left(f\right)\right)\leq c_{46}\Omega_{M,\omega}^{k}\left(\delta,f\right). \qquad (2.13)$$

Now combining (2.12) and (2.13) we obtain the inequality (1.5) of Theorem 1.3.

Proof of Theorem 1.5. Let $f\in L_{M}\left(\mathrm{G},\omega\right)$. Then by virtue of Theorem 2.2 the operartor $A:E_{M}\left(D,\omega\right)\rightarrow E_{M}\left(G,\omega\right)$ is bounded one-to-one and onto and $A\left(f_{0}^{+}\right)=f$. The function f has the followinf Faber series:

$$f\left(z\right)\sim\sum_{m=0}^{\infty}a_{m}\left(f\right)\phi_{m}\left(z\right).$$

Using Lemma 1 in [18] we conclude that $f_{0}^{+}\in E_{M}\left(D,\omega\right)$. Then for the function f_{0}^{+} the Taylor

$$\sum_{m=0}^{\infty}a_{m}\left(f\right)w^{m}.$$

expansion holds. Note that $f_{0}^{+}\in E^{1}\left(D\right)$. Then boundary function $f_{0}^{+}\in L_{M}\left(T,\omega\right)$. By [13, Theorem 3.4] for the function f_{0}^{+} we have the following Fourier expansion:

$$f_{0}^{+}\left(w\right)\sim\sum_{m=0}^{\infty}a_{m}\left(f\right)e^{imw}.$$

According to Therem 2.1 the linear operatör $A: E_M(D,\omega) \to E_M(G,\omega)$ is bounded. Then if we consider boundedness of the operatör $A: E_M(D,\omega) \to E_M(G,\omega)$ and Theorem 1.3 we have

$$\begin{aligned} &\| f - U_n(\cdot, f)\|_{L_M(T,\omega)} \\ &= \left\| A(f_0^+) - A(U_n(\cdot, f_0^+))\right\|_{L_M(\Gamma,\omega)} \le c_{47} \| f_0^+ - U_n(\cdot, f_0^+)\|_{L_M(T,\omega_0)} \\ &\le c_{48}\Omega^k_{M,\omega_0}\left(\frac{1}{n}, f_0^+\right) = c_{49}\Omega^k_{\Gamma,M,\omega}\left(\frac{1}{n}, f\right). \end{aligned}$$

.

which completes the proof of inequality (1.10)

Remark 3.1. Let $L_M(T,\omega)$ be the weighted Orlicz space with Boyd indices $0 < \alpha_M \le \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. Then by virtue of Theorem 4 in [18] for $f \in L_M(T,\omega)$ the inequality

$$\Omega^k_{M,\omega}\left(\frac{1}{n}, f\right) \le c_{50} n^{-2k} \left\{ E_0(f)_{M,\omega} + \sum_{m=1}^{n} m^{2k-1} E_m(f)_{M,\omega} \right\} \tag{2.14}$$

holds with a constant c_{50} independent of n. If the summability method with the matrix Λ satisfy the condition $(b_{k,M})$ or $(b^*_{k,M})$, then for $f \in L_M(T,\omega)$ relation (1.5) and inequality (2.14) immediately yield

$$\| f - U_n(\cdot, f)\|_{L_M(T,\omega)} \le c_{51} n^{-2k} \left\{ E_0(f)_{M,\omega} + \sum_{m=1}^{n} m^{2k-1} E_m(f)_{M,\omega} \right\}. \tag{2.15}$$

The inequality (2.15) holds for Zygmund-Riesz means of order k. Note that in the Lebesgue space $L_p(T)$, $1 < p < \infty$ the inequality (2.15) was proved in [40].

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REFERENCES

1. Akgün R., Israfilov D. M., Approximation by interpolating polynomials in Smirnov-Orlicz clas, J. Korean Math. Soc., 2006, 43 , pp.2413-424.

2. Akgün R., Israfilov D. M., Simultaneous and converse approximation theorems in weighted Orlicz spaces, Bull. Belg. Math. Soc. Simon Stevin, 2010,17, pp.13-28.
3. Akgun R., Improved converse theorems and fractional moduli of smoothness in Orlicz spaces, Bull. Malays. Math. Soc. 2013, (2) 36 (1) , pp.49-62.
4. Aliev F.A., Velieva N.I. LMI method for solving the inverse problem of optimization of a discrete periodic system by output variable, Reports of AS Azerb. 2007, V.63,N. 4, pp.24-31 (in Russian).
5. Aliev F.A., Larin V.B., Velieva N.I., Gasimova K.G., Farajova Sh. A. LMI method for solving bhh equations, Control and Optimization with Industrial Applications, 2020, pp.95-99
6. Bennett C., Sharpley Yu. I., Interpolation of operators, Academic Press, 1988.
7. A. Böttcher A., Karlovich Yu. I., Carleson curves, Muckenhoupt weights and Teoplitz operators, Birkhauser-Verlag, 1997.
8. Boyd D. W., Spaces between a par of reflexive Lebesgue spaces, Proc. Amer. Math. Soc. 1967,18, pp.215-219.
9. Boyd D. W., Indices of function spaces and their relationship to interpolation, Canad. J. Math. 1969, 21, pp.1245-1254.
10. Boyd D.W., Indices fort he Orlicz spaces, Pacific J. Math. 38 (1971), 315-325.
11. Cao, J. D., Stechkin inequalities for summability methods, Internat. J. Math. Sci., 1997, 20, pp.93-100.
12. Chikina T. S., Approximation by Zygmund-Riesz means in the p-variation metric, Anal. Math. 2013,39 (1), pp.29-44.
13. Duren P. L., Theory of H^p spaces, Academic Press 1970.
14. De Vore R. A., Lorentz G. G., Constructive Approximation, Springer, 1993.
15. Guven A., Israfilov D. M., Approximation by means of Fourier trigonometric series in weighted Orlicz spaces, Adv. Stud. Contemp. Math (Kyungshang), 2009, 19 (2), 283-295.
16. Goluzin G. M., Geometric theory of functions of a complex variable, Translation of Mathematical Monographs, 26, Providence, PI: AMS 1968.
17. Gavrilyuk V. G., Linear methods of summing Fourier series and best approximation, Ukr. Math. J, 1963, 15 (4), pp.412-417.
18. Israfilov D. M., Akgün, R., Approximation in weighted Smirnov-Orlicz classes, Journal of Mathematics of Kyoto University, 46 (4) (2006), 755-770.
19. Israfilov D. M., Guven A., Approximation in weighted Smirnov classes, East J. Approx. 2005,11, pp.91-102.
20. Israfilov D. M., Guven A., Approximation by trigonometric polynomials in weighted Orlicz spaces, Studia Math. 2006,174 (2), pp.147-168.
21. Il'yasov N. A., Approximation of periodic functions by Zygmund means, Mat. Zametki 1986, 39 (3) pp.367-382 (in Russian).
22. Jafarov S. Z., Approximation by Fejer sums of Fourier trigonometric series in weighted Orlicz spacers, Hacet. J. of Math. Stat. 2013, 42 (3), pp.259-268.

23. Jafarov S. Z., Linear methods for summing Fourier series and approximation in weighted Lebesgue spaces with variable exponents, Ukrainian Mathematical Journal, 2015, V. 66, No. 10, March, pp.1509-1518,
24. Jafarov S. Z., Linear methods of summing Fourier series and approximation in weighted Orlicz spaces, Turkish Journal of Math., 2018, 42, pp.2916-2925.
25. Jafarov S. Z., Estimates of the approximation by Zygmund sums in Morrey-Smirnov classes of analytic functions, Azerbaijan Journal of Mathematics, 2020, 10 (2), pp.110-124.
26. Krasnosel’skii M. A., Rutickiĭ Ya. B., Convex Function and Orlicz Spaces, Noordhoff, 1961.
27. Karlovich A.Yu., Algebras of singular integral operators with piecewise continuous coefficients on reflexive Orlicz spaces, Math. Nachr. 1996,179, pp.187-222.
28. Karlovich A.Yu., Singular integral operators with piecewise continuous coefficients in reflexive rearrangement-invariant spaces, Integral. Equations and Oper. Th. 1998,32, pp.436-481.
29. Karlovich A. Yu., Algebras of singular integral operators with PC coefficients in rearrangement invariant spaces, with Muckenhoupt weights, J. Operator Theory 2002,47, pp.303-323.
30. Karlovich, A.Yu., Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces, J. Integal Equations . Appl. 2003, 15, pp.263-320.
31. Kokilashvili V. M., On analytic functions of Smirnov-Orlicz classes, Studia Math. 1968,31 ,pp. 43-59.
32. Kokilashvili V. M., Samko S. G., Operators of harmonic analysis in weighted spaces with non-standart growth, J. Math. Anal. Appl. 2009, 352, pp.15-34.
33. Kokilashvili V. M., Tsanava Ts., Approximation by linear summability means in weighted variable exponent Lebesgue spaces, Proc. A. Razmadze Math. Inst. 2010,154, pp.147-150.
34. Matuszewska W., Orlicz, W., On certain properties of ϕ-functions, Bull. Acad. Polon. Sci., Ser. Math. Aster. Et Phs1969,. 8, 7 , pp.439-443.
35. Muckenhoupt B., Weighted norm inequalities fort he Hardy maximal function, Trans. Amer. Math. Soc. 1972,165 , pp.207-226.
36. Maligranda L., Indices and interpolation, Dissertationes Math. 1985,234p.
37. Pommerenke Ch., Boundary Behavior of Conformal Maps, Berlin, Springer-Verlag 1992.
38. Ramazanov A.-R. K., On approximation by polynomials and rational functions in Orlicz spaces, Anal. Math. 1984, 10, pp.117-132.
39. Rao M. M., Ren Z. D.,Theory of Orlicz space, Marcel Dekker, New York, 1991.
40. Suetin P. K., Series of Faber polynomials, Gordon and Breach, 1, Reading, 1998.

41. Stechkin S. B., The approximation of periodicfunctions by Fejer sums, Trudy Math. Inst. Steklov, 62, 1961, pp.48-60 (in Russian).
42. Timan A. F., Best approximation of a function and linear methods of summing Fourier series, Izv. Akad. Nauk SSSR ser. Math. 1965,29, 587-604 (in Russian).
43. Warschawskii S. E., Über das Ranverhalten der Ableitung der Abbildungs-functionen bei konformer Abbildung. Math. Z., 1932, 35, pp.321-456.