NOTES ON SOME SIMPSON TYPE INTEGRAL INEQUALITIES FOR 
s-GEOMETRICALLY CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In the paper, the authors present several integral inequalities of the Hermite-Hadamard type for s-geometrically convex functions and apply these new integral inequalities to correct several errors appeared in [2].

Keywords: integral inequality, Hermite-Hadamard type, s-geometrically convex function, error, correction, Hölder’s integral inequality.

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1. Introduction

The concept of s-geometrically functions was introduced in [6, Definition 1.9].

Definition 1 ([6, Definition 1.9]). A function \( f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}_+ \) is said to be an s-geometrically convex function for some \( s \in (0,1] \), if the inequality

\[
 f(x^t y^{1-t}) \leq [f(x)]^t [f(y)]^{1-t}
\]

holds for all \( x, y \in I \) and \( t \in [0,1] \).

Remark 1. Let \( s \in (0,1] \) and let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+ \) be an s-geometrically convex function.

(1) If \( s = 1 \), the s-geometrically convex function becomes a geometrically convex function on \( \mathbb{R}_+ \).

(2) If \( s \in (0,1) \), then \( f(x) \geq 1 \) is valid for all \( x \in I \).

In the paper [3], an integral identity was created as follows.
Lemma 1 ([3, Lemma 1]). Let \( I \subseteq \mathbb{R} \) and let \( f : I \to \mathbb{R} \) be differentiable on \( I^\circ \) such that \( f' \in L_1([a,b]) \), where \( a, b \in I \) with \( a < b \). Then
\[
\frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx
\]
\[
= \frac{b-a}{2} \int_0^1 \left( t - \frac{1}{3} \right) \left[ f'(\frac{1-t}{2}a + \frac{1+t}{2}b) - f'(\frac{1+t}{2}a + \frac{1-t}{2}b) \right] dt.
\]

In view of Lemma 1, the authors of the paper [2] established the following Hermite-Hadamard type inequalities for s-geometrically convex functions.

Theorem 1 ([2, Theorems 2.2 and 2.4]). Let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R} \) be a differentiable mapping on \( I^\circ \) such that \( f' \in L_1([a,b]) \), where \( a, b \in I^\circ \) with \( a < b \). If \( |f'(x)|^q \) is s-geometrically convex and decreasing on \([a,b]\) for \( s \in (0,1) \) and \( q \geq 1 \), then
\[
\left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right|
\]
\[
\leq \frac{b-a}{2} \left( \frac{5}{36} \right)^{1-1/q} \left\{ h_1\left( \alpha\left(\frac{sq}{2}, \frac{sq}{2}\right) \right) \right\}^{1/q} + \left\{ h_2\left( \alpha\left(\frac{sq}{2}, \frac{sq}{2}\right) \right) \right\}^{1/q}
\]
\[
\times \begin{cases} 
|f'(a)f'(b)|^{s/2}, & |f'(a)| \leq 1; \\
|f'(a)|^{1-s/2}|f'(b)|^{s/2}, & |f'(b)| \leq 1 \leq |f'(a)|; \\
|f'(a)f'(b)|^{1-s/2}, & |f'(b)| \geq 1,
\end{cases}
\]
where \( \alpha(u,v) = \frac{|f'(b)|^u}{|f'(a)|^v} \) for \( u, v > 0 \),
\[
\begin{align*}
h_1(\alpha) &= \begin{cases} 
\frac{5}{36}, & \alpha = 1; \\
\frac{6\alpha^{2/3} + (\alpha - 2)\ln \alpha - 3\alpha - 3}{6(\ln \alpha)^2}, & \alpha \neq 1,
\end{cases} \\
h_2(\alpha) &= \begin{cases} 
\frac{5}{36}, & \alpha = 1; \\
\frac{6/\alpha^{2/3} + (2 - 1/\alpha)\ln \alpha - 3/\alpha - 3}{6(\ln \alpha)^2}, & \alpha \neq 1,
\end{cases}
\end{align*}
\]
Theorem 2 ([2, Theorem 2.3]). Let \( f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^\circ \) such that \( f' \in L_1([a,b]) \), where \( a, b \in I^\circ \) with \( a < b \). If \( |f'(x)|^q \) is s-geometrically convex and decreasing on \( [a,b] \) for \( s \in (0,1) \) and \( q > 1 \) with \( \frac{1}{q} + \frac{1}{p} = 1 \), then

\[
\left| \frac{1}{6} f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| 
\leq \frac{b-a}{2} \left( \frac{2(1+2^{p+1})}{6^{p+1}(p+1)} \right)^{1/p} \left\{ h_3\left( \frac{2s}{2}, \frac{2s}{2} \right) \right\}^{1/q} + \left\{ h_4\left( \frac{2s}{2}, \frac{2s}{2} \right) \right\}^{1/q} 
\leq \frac{b-a}{2} \left( \frac{2(1+2^{p+1})}{6^{p+1}(p+1)} \right)^{1/p} \left\{ \frac{1}{2} \right\}^{1/q} \left\{ f'(a) f'(b) \right\}^{s/2} , \quad |f'(a)| \leq 1; 
\times \left\{ \frac{1}{2} \right\}^{1-s/2} \left| f'(b) \right|^{s/2} , \quad |f'(b)| \leq 1 \leq |f'(a)|, 
\left\{ \frac{1}{2} \right\}^{1-s/2} \left| f'(a) f'(b) \right|^{s/2} , \quad |f'(b)| \geq 1,
\right.
\]

where \( \alpha(u,v) = \frac{|f'(b)|^u}{|f'(a)|^v} \) for \( u, v > 0 \),

\[
h_3(\alpha) = \begin{cases} 
1, & \alpha = 1; \\
\frac{\alpha - 1}{\ln \alpha}, & \alpha \neq 1,
\end{cases} \quad \text{and} \quad h_4(\alpha) = \begin{cases} 
1, & \alpha = 1; \\
\frac{\alpha - 1}{\alpha} \left( \frac{\alpha}{2} \right), & \alpha \neq 1.
\end{cases}
\]

Remark 2. Under the conditions of Theorems 1 and 2,

1. if \( q = 1 \), Theorem 1 is just [2, Theorem 2.2];
2. if \( q > 1 \), Theorem 1 is equivalent to [2, Theorem 2.4];
3. for \( \alpha > 0 \), the relations \( h_2(\frac{1}{\alpha a}) = h_1(\frac{1}{a}) \) and \( h_4(\frac{1}{\alpha a}) = h_3(\frac{1}{a}) \) are valid;

We claim that there existed heavy errors and serious mistakes not only in Theorems 1 and 2 but also in other propositions in the paper [2].

In this paper, we will correct, as done in the papers [4, 5], those heavy errors and serious mistakes appeared in Theorems 1 and 2 and other propositions in the paper [2], by establishing several new integral inequalities of the Hermite-Hadamard type for s-geometrically convex functions.

2. Corrected versions of Theorems 1 and 2 in the paper [2]
Now we start out to correct the errors and mistakes in Theorems 1 and 2 by establishing several new integral inequalities of the Hermite-Hadamard type for \( s \)-geometrically convex functions.

**Theorem 3** (Corrected version of Theorem 1). Let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R} \) be a differentiable mapping on \( I^+ \), let \( a,b \in I^+ \) with \( a < b \), and let \( f' \in L_1([a,b]) \). If \( |f'(x)|^q \) is \( s \)-geometrically convex and decreasing on \([a,b]\) for \( q \geq 1 \) and \( s \in (0,1] \), then

\[
\left| \frac{1}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} \left[ \frac{5}{36} \right]^{1-1/q}
\]

\[
\times \left[ h_1\left( \alpha\left( \frac{sq}{2}, \frac{sq}{2} \right) \right) \right]^{1/q} \left[ h_1\left( \frac{1}{\alpha(sq/2, sq/2)} \right) \right]^{1/q} \| f'(a) f'(b) \|^{-s/2},
\]

where \( \alpha(u,v) \) and \( h_1(\alpha) \) are defined as in Theorem 1.

Proof. From Lemma 1 and Hölder's integral inequality, we obtain

\[
\left| \frac{1}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| 
\]

\[
\leq \frac{b-a}{2} \left[ \int_0^1 \left| \frac{t}{2} \left( 1-t \right) a + \frac{1+t}{2} b \right| + \left| \frac{1}{2} \left( 1+t \right) a + \frac{1-t}{2} b \right| \right] \, dt
\]

\[
\leq \frac{b-a}{2} \left[ \int_0^1 \left| \frac{t}{2} \right| - \frac{1}{3} \, dt \right]^{1-1/q} \left[ \left( \int_0^1 \left| \frac{t}{2} \right| - \frac{1}{3} \, dt \right) \left| f'(\frac{1-t}{2} a + \frac{1+t}{2} b) \right|^q \right]^{1/q}
\]

\[
+ \left( \int_0^1 \left| \frac{t}{2} \right| - \frac{1}{3} \, dt \right) \left| f'(\frac{1+t}{2} a + \frac{1-t}{2} b) \right|^q \right]^{1/q}.
\]

Let \( 0 < \mu \leq 1 \leq \eta \) and \( 0 < s, t \leq 1 \). Then it was deduced in [1, p.4] that

\[
\mu^{\nu} \leq \mu^s \quad \text{and} \quad \eta^{r^s} \leq \eta^{s+1-s}.
\]

Considering the condition that \( |f''|^q \) is decreasing and \( s \)-geometrically convex on \([a,b]\) and making use of the inequalities in (3) yield

\[
\left| f'(\frac{1-t}{2} a + \frac{1+t}{2} b) \right|^q \leq \left| f'(a^{(1-t)/2} b^{(1+t)/2}) \right|^q
\]
\[
\left| f'(a)^{q(1-\gamma)/2} f''(b) \right|^{q(1-\gamma)/2} \leq \left| f'(a)^{q(1-\gamma)/2} f''(b) \right|^{q(1-\gamma)/2} \leq \left| f'(a)^{q(1-\gamma)/2} f''(b) \right|^{q(1-\gamma)/2}
\]

and
\[
\left| f'(\frac{1+t}{2}a + \frac{1-t}{2}b) \right|^{q/2} \leq \left| f'(a)^{q(1-\gamma)/2} f''(b) \right|^{q(1-\gamma)/2} \left| f'(a)^{q(1-\gamma)/2} f''(b) \right|^{q(1-\gamma)/2}.
\]

Similarly or straightforwardly, we acquire
\[
\int_0^1 \left| t - \frac{1}{3} \right| dt = \frac{5}{36},
\]
\[
\int_0^1 \left| t - \frac{1}{3} \right| f'(\frac{1-t}{2}a + \frac{1+t}{2}b) \right|^{q/2} dt \leq \left| f'(a)^{q(1-\gamma)/2} f''(b) \right|^{q(1-\gamma)/2} \int_0^1 \left| t - \frac{1}{3} \right| \left| f'(b) \right|^{q(1-\gamma)/2} \left| f'(a)^{q(1-\gamma)/2} f''(b) \right|^{q(1-\gamma)/2} dt
\]
\[
= \left| f'(a)^{q(1-\gamma)/2} f''(b) \right|^{q(1-\gamma)/2} h_1 \left( \frac{\alpha(s, s/2)}{\alpha(s/2, s/2)} \right),
\]
\[
\int_0^1 \left| t - \frac{1}{3} \right| f'(\frac{1+t}{2}a + \frac{1-t}{2}b) \right|^{q/2} dt \leq \left| f'(a)^{q(1-\gamma)/2} f''(b) \right|^{q(1-\gamma)/2} h_1 \left( \frac{1}{\alpha(s/2, s/2)} \right).
\]

Substituting the inequalities (4) and (5) into the inequality (2) and simplifying result in the inequality (1). Theorem 3 is thus proved.

**Corollary 1.** Under the conditions of Theorem 3, if \( q=1 \), then
\[
\left| \frac{1}{6} f(a) + 4 f\left( \frac{a+b}{2} \right) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{b-a}{2} \left[ h_1 \left( \alpha \left( \frac{s}{2}, \frac{s}{2} \right) \right) + h_1 \left( \frac{1}{\alpha(s/2, s/2)} \right) \right] \left| f'(a)^{1-\gamma/2} \right|,
\]
where \( \alpha(u, v) \) and \( h_1(\alpha) \) are defined as in Theorem 1.

**Corollary 2.** Under the conditions of Theorem 3, if \( s=1 \), then
\[
\left| \frac{1}{6} f(a) + 4 f\left( \frac{a+b}{2} \right) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{5}{36} \left( \left[ h_1 \left( \alpha \left( \frac{q}{2}, \frac{q}{2} \right) \right) \right]^{1/q} + \left[ h_1 \left( \frac{1}{\alpha(q/2, q/2)} \right) \right]^{1/q} \right)^{1/q} \left| f'(a)^{1/2} \right|,
\]
where \( \alpha(u, v) \) and \( h_1(\alpha) \) are defined as in Theorem 1.
By virtue of the same ideas and approaches as in the proof of Theorem 3, we can find out the following results.

**Theorem 4** (Corrected version of Theorem 2). Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on $I$ such that $f' \in L_1([a,b])$, where $a, b \in I$ with $a < b$. If $\left| f'(x) \right|^q$ is s-geometrically convex and decreasing on $[a,b]$ for $s \in (0,1]$ and $q > 1$ with $\frac{1}{q} + \frac{1}{p} = 1$, then

$$
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} \left[ \frac{2(1+2^{p+1})}{6^{p+1}(p+1)} \right]^{1/p} \\
\times \left\{ h_3 \left( \alpha \left( \frac{s}{2}, \frac{s}{2} \right) \right) \right\}^{1/q} + \left\{ h_3 \left( \frac{1}{\alpha(s/2,s/2)} \right) \right\}^{1/q} \left| f'(a) f'(b) \right|^{1-s/2},
$$

where the function $\alpha(u,v)$ is defined as in Theorem 1 and the function $h_3(\alpha)$ is defined as in Theorem 2.

**Corollary 3.** Under the conditions of Theorem 3, if $s=1$, then

$$
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} \left[ \frac{2(1+2^{p+1})}{6^{p+1}(p+1)} \right]^{1/p} \\
\times \left\{ h_3 \left( \alpha \left( \frac{q}{2}, \frac{q}{2} \right) \right) \right\}^{1/q} + \left\{ h_3 \left( \frac{1}{\alpha(q/2,q/2)} \right) \right\}^{1/q} \left| f'(a) f'(b) \right|^{1/2},
$$

where the function $\alpha(u,v)$ is defined as in Theorem 1 and the function $h_3(\alpha)$ is defined as in Theorem 2.

### 3. Corrected versions of three propositions in the paper [2]

In this section, we will apply several integral inequalities of the Hermite-Hadamard type for s-geometrically convex functions to construct some inequalities for means.

For two positive numbers $a > 0$ and $b > 0$, define

$$ A(a,b) = \frac{a+b}{2}, \quad G(a,b) = \sqrt{ab}, $$

$$ M_2(a,b) = G(a,b) - \frac{G(a+b)}{2}, \quad M_3(a,b) = G(a,b) - \frac{G(a+b)}{3}. $$

...
and

$$L_p(a, b) = \begin{cases} \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{1/p}, & a \neq b, p \neq 0, -1; \\ a, & a = b. \end{cases}$$

These means are respectively called the arithmetic mean, the geometric mean, and the generalized logarithmic mean of two positive numbers $a > 0$ and $b > 0$.

Let $f(x) = \frac{x^s}{s}$ for $x > 0$, $0 < s < 1$, and $q > 0$. Then the function $|f'(x)| = x^{s-1}$ is geometrically convex on $x \in (0, 1]$. Since the inequality

$$\left| f'(x^t y^{1-t}) \right| = \left( x^{s-1} \right)^t \left( y^{s-1} \right)^{1-t} \leq \left( x^{s-1} \right)^t \left( y^{s-1} \right)^{1-t} = \left| f'(x) \right|^t \left| f'(y) \right|^{1-t}$$

holds for all $x, y \in (0, 1]$ and $t \in [0, 1]$, so the function $|f''(x)|^q = x^{(s-1)q}$ is $s$-geometrically convex in $x \in (0, 1]$.

**Theorem 5.** Let $0 < a < b \leq 1$ and $0 < s < 1$. Then

$$\left| \frac{2A(a^s, b^s) + 4[A(a, b)]^s}{6} - [L_s(a, b)]^s \right| \leq \frac{(b-a)s}{2} \left[ G(a, b) \right]^{s-1} \left[ h_s(b(a, b)) + h_s(b(b, a)) \right]$$

where $\beta(a, b) = \left( \frac{b}{a} \right)^{(s-1)/2}$ and $h_s(\alpha)$ is defined as in Theorem 1.

Proof. Using Lemma 1, we obtain
$$\frac{1}{s} \left| A(a^s, b^s) + 2[A(a, b)]^s - [L_s(a, b)]^s \right|$$

$$\leq \frac{b-a}{2} \int_0^1 t - \frac{1}{3} \left[ f''\left(\frac{1-t}{2} a + \frac{1+t}{2} b\right) + f''\left(\frac{1+t}{2} a + \frac{1-t}{2} b\right) \right] dt$$

$$= \frac{b-a}{2} \int_0^1 t - \frac{1}{3} \left[ \left(\frac{1-t}{2} a + \frac{1+t}{2} b\right)^{s-1} + \left(\frac{1+t}{2} a + \frac{1-t}{2} b\right)^{s-1} \right] dt$$

$$\leq \frac{b-a}{2} \int_0^1 t - \frac{1}{3} \left[ \left(a^{(1-t)/2} b^{(1+t)/2}\right)^{s-1} + \left(a^{(1+t)/2} b^{(1-t)/2}\right)^{s-1} \right] dt$$

$$= \frac{b-a}{2} \left[ G(a, b) \right]^{s-1}$$

$$= \frac{b-a}{2} \left[ h_1(\beta(a, b)) + h_1(\beta(b, a)) \right].$$

Theorem 5 is thus proved.

**Corollary 4.** Let $0 < a < b \leq 1$ and $0 < s < 1$. Then

$$\left| 2\frac{A(a^s, b^s) + 4[A(a, b)]^s - [L_s(a, b)]^s}{6} \right| \leq \frac{5(b-a)s}{36} \cdot A(a^{s-1}, b^{s-1}).$$

Proof. By the inequality (7) and the geometric-arithmetic mean inequality, we have

$$\frac{1}{s} \left| A(a^s, b^s) + 2[A(a, b)]^s - [L_s(a, b)]^s \right|$$

$$\leq \frac{b-a}{2} \int_0^1 t - \frac{1}{3} \left[ \left(a^{(1-t)/2} b^{(1+t)/2}\right)^{s-1} + \left(a^{(1+t)/2} b^{(1-t)/2}\right)^{s-1} \right] dt$$

$$\leq \frac{b-a}{2} \int_0^1 t - \frac{1}{3} \left[ \left(\frac{1-t}{2} a^{s-1} + \frac{1+t}{2} b^{s-1}\right) + \left(\frac{1+t}{2} a^{s-1} + \frac{1-t}{2} b^{s-1}\right) \right] dt$$

$$= (b-a) A(a^{s-1}, b^{s-1}) \int_0^1 t - \frac{1}{3} dt$$

$$= \frac{5(b-a)}{36} A(a^{s-1}, b^{s-1}).$$

Corollary 4 is thus proved.

Under the conditions of Theorem 5, from the inequalities (6) and (7), it follows that
Therefore, we can correct [2, Propositions 3.1, 3.2, and 3.3] as follows.

**Corollary 5** (Corrected version of [2, Propositions 3.1 and 3.3]). Let \(0 < a < b \leq 1\), \(0 < s < 1\), and \(q \geq 1\). Then

\[
\frac{1}{6} \left[ \frac{2A(a^s, b^s) + 4[A(a, b)]^s}{s} - \frac{[L_s(a, b)]^s}{s} \right] \leq \frac{b - a}{2} \left( \frac{5}{36} \right)^{1-1/q}
\]

\[
\times G(a^{(s-1)(2-s)}, b^{(s-1)(2-s)}) \left\{ h_1 \left( \frac{\alpha(\frac{sq}{2}, \frac{sq}{2})}{\alpha(\frac{sq}{2}, \frac{sq}{2})} \right) \right\}^{1/q} + h_1 \left( \frac{1}{\alpha(sq/2, sq/2)} \right) \right\}^{1/q}
\]

where \(h_1(\alpha)\) is defined as in Theorem 1 and \(\alpha(u, v) = \frac{b^{(s-1)v}}{a^{(s-1)u}}\) for \(u, v > 0\).

**Corollary 6** (Corrected version of [2, Propositions 3.2]). Let \(0 < a < b \leq 1\), \(0 < s < 1\), and \(q > 1\) with \(\frac{1}{q} + \frac{1}{p} = 1\). Then

\[
\frac{1}{6} \left[ \frac{2A(a^s, b^s) + 4[A(a, b)]^s}{s} - \frac{[L_s(a, b)]^s}{s} \right] \leq \frac{b - a}{2} \left( \frac{2(1 + 2^{p+1})}{6^{p+1}(p + 1)} \right)^{1-1/q}
\]

\[
\times G(a^{(s-1)(2-s)}, b^{(s-1)(2-s)}) \left\{ h_3 \left( \frac{\alpha(\frac{sq}{2}, \frac{sq}{2})}{\alpha(\frac{sq}{2}, \frac{sq}{2})} \right) \right\}^{1/q} + h_3 \left( \frac{1}{\alpha(sq/2, sq/2)} \right) \right\}^{1/q}
\]

where \(h_3(\alpha)\) is defined as in Theorem 2 and \(\alpha(u, v) = \frac{b^{(s-1)v}}{a^{(s-1)u}}\) for \(u, v > 0\).

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