A SYMBOLIC ALGORITHM FOR POLYNOMIAL INTERPOLATION WITH STIELTJES CONDITIONS IN MAPLE

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Abstract. In this paper, we present a Maple implementation of a symbolic algorithm for polynomial interpolation with the combination of general, differential and integral conditions, so-called Stieltjes conditions at arbitrary points. This algorithm will be helpful to implement this in commercial software packages such as Mathematica, Matlab, Singular, SCIlab etc. Sample computations are presented to illustrate the Maple package.

Keywords: Polynomial interpolation, integral conditions, error estimation, Stieltjes conditions, Symbolic Algorithm.

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1. Introduction.

Most of the time, scientists and researchers often come up with data points in science and engineering, obtained by sampling or experimentation, which represent the values of a function for a limited number of values of the independent variable. It is often required to interpolate the value of that function for an intermediate value of the independent variable. There exist many interpolation techniques in the literature for general functional values. In this paper, we present a Maple package for an algorithm of a polynomial interpolation, presented in [5], with a finite set of integral conditions alone as well as the combination of general, differential and integral conditions, so-called Stieltjes conditions. For simplicity, recall the algorithm for polynomial interpolation, see [5] for further details. Various symbolic methods for interpolations are in the literature, see, for example [2, 5, 6, 7, 8, 9, 11, 12, 13, 10,14].

2. Symbolic Algorithm.

The general form of the interpolation problem is defined as follows [1, 4, 10, 14]: Suppose $\mathcal{S}$ is a normed linear space. For a finite linearly independent set $\Theta \subset \mathcal{S}$ of bounded functionals and associated values $\Omega = \{\alpha_\theta : \theta \in \Theta\} \subset \mathbb{R}$, the *interpolation problem* is to find a $\tilde{f}_s(x) \in \mathcal{S}$ such that

$$\Theta(\tilde{f}_s) = \Omega, \text{ i.e. } \theta \tilde{f}_s = \alpha_\theta,\ \theta \in \Theta. \tag{1}$$

Here 's' is called the order of the interpolating function $\tilde{f}_s(x) \in \mathcal{S}$. To describe the polynomial interpolation, let $\mathcal{S} = \mathbb{K}[x]$ be a polynomial ring over a field $\mathbb{K}$, where $\mathbb{K} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

Definition 1: [5] We call the pair (Θ, Ω) a *polynomial interpolation problem*, where Θ is a finite linearly independent set of functionals with associated values $\Omega \subset \mathbb{K}$.

Definition 2: [5] A polynomial interpolation problem (Θ, Ω) is called *regular* with respected to Θ if (Θ, Ω) has a unique solution for each choice of values of $\Omega \subset \mathbb{K}$ such that $\Theta(\tilde{f}_s) = \Omega$. Otherwise, it is called *singular*.

The following proposition gives the regularity test in terms of linear algebra.

Proposition 1: [7, 14] Let $M = \{m_0, \ldots, m_{t-1}\}$ be a basis for $\mathcal{M}$, a finite dimensional subspace of $\mathcal{S}$, and $\Theta = \{\theta_0, \ldots, \theta_{s-1}\} \subset \mathcal{S}^*$ with θ_i linearly independent. Then the following statements are equivalent:

(i) The polynomial interpolation problem is regular for $\mathcal{M}$ with respected to Θ.

(ii) $t = s$, and the *evaluation matrix*,

$$\Theta M = \begin{pmatrix} \theta_0(m_0) & \cdots & \theta_0(m_{t-1}) \\ \vdots & \ddots & \vdots \\ \theta_{s-1}(m_0) & \cdots & \theta_{s-1}(m_{t-1}) \end{pmatrix} \tag{2}$$

is nonsingular. Denote the evaluation matrix ΘM by $\mathcal{E}$ for simplicity.

(iii) $\mathcal{S} = M \oplus \Theta^{\perp}$.

The following theorem gives the main algorithm for constructing a polynomial interpolation with Stieltjes conditions at arbitrary points.

Theorem 1: Let $\Theta = \{\theta_0, \ldots, \theta_{s-1}\}$ be a finite set of the form $\{E_c A : c \in \mathbb{R}\} \subset \mathcal{S}^*$ with associated values $\Omega = \{\alpha_{\theta_i} : \theta_i \in \Theta\} \subset \mathbb{R}$. Then there exists a unique polynomial $\tilde{f}_s(x) = a_0 + a_1 x + \cdots + a_s x^s$ satisfying Θ if and only if the evaluation matrix $\mathcal{E}$ is regular, and $\tilde{f}_s(x)$ is given by,

$$\tilde{f}_s(x) = a_0 + (2\tilde{a}_1)x + \cdots + (s\tilde{a}_{s-1}), \tag{3}$$

where $\tilde{a}_i = \dfrac{a_i}{i+1}$.

Proof: See [5, Theorem 2].

The following section presets the main content of the paper, namely the Maple implementation of the algorithm presented in Theorem 4.

3. Maple Implementation.

The algorithm for polynomial interpolation with Stieltjes conditions at arbitrary points is implemented in Maple by creating different data types. We use the Maple package `IntDiffOp` implemented by Anja Korporal et al. [3] to express the Stieltjes conditions operators.

Using `IntDiffOp` package, we have the Stieltjes conditions as follows. In the following syntax, `evp`, `a` and `b` are the evaluation, differential and integral points respectively.

```
> BOUNDOP(EVOP(evp, EVDIFFOP(a), EVINTOP(EVINTTERM(b))))
```

The following procedure gives evaluation matrix as given in equation (2). In the following procedure `SC` is Stieltjes conditions matrix.

```
> with(IntDiffOp):
> EvaluationMat := proc (SC::Matrix)
> local r, c, elts, fs;
> r, c := LinearAlgebra[Dimension](SC);
> fs := Matrix(1, r, [seq(x^(i-1), i=1..r)]);
> elts:=seq(seq(ApplyOperator(SC[t,1],fs[1,j]),j=1..r),t=1..r);
> return Matrix(r, r, [elts])
> end proc:
```

Using the following procedure `PolynomialInterpolation(SC,CM)`, one can obtain the polynomial interpolation, where SC is Stieltjes conditions matrix and `CM` is associated values.

```
> PolynomialInterpolation:=proc(SC::Matrix,CM::Matrix)
> local r, c, fs, evm, invevm, approx;
> r, c := LinearAlgebra[Dimension](SC);
> fs := Matrix(1, r, [seq(x^(i-1), i = 1..r)]);
> evm := EvaluationMat(SC);
> invevm := 1/evm;
> approx := fs.invevm.CM;
> return simplify(approx[1, 1])
> end proc:
```

Sample computations are presented in the following section using the maple implementation.

4. Sample Computations.

Example 1: Consider the following Stieltjes conditions (the combination of general, differential and integral conditions) at arbitrary points.

$$f(0)=1,\ f(0.2)=3, \left(\frac{d}{dx} f(x)\right)_{x=0.3}=4,\ f(0.3)+\int_0^{0.3} f(x)\,dx=5,$$
$$\left(\frac{d}{dx} f(x)\right)_{x=0.4}=6, \left(\frac{d}{dx} f(x)\right)_{x=0.5}+\int_0^{0.5} f(x)\,dx=7,\ f(0.6)=9,$$
$$f(0.8)+\int_0^{0.8} f(x)\,dx=13, \left(\frac{d}{dx} f(x)\right)_{x=0.9}=15,\ f(1.0)=16.$$

Using the Maple implementation, we compute the polynomial which satisfying the given Stieltjes conditions.

```
> C1:=BOUNDOP(EVOP(0,EVDIFFOP(1),EVINTOP(EVINTTERM(0,1)))):
> c1:=1:
> C2:=BOUNDOP(EVOP(0.2,EVDIFFOP(1),EVINTOP(EVINTTERM(0,1)))):
> c2 := 3;:
> C3:=BOUNDOP(EVOP(0.3,EVDIFFOP(0,1),EVINTOP(EVINTTERM(0,1)))):
> c3 := 4:
> C4:=BOUNDOP(EVOP(0.3,EVDIFFOP(1),EVINTOP(EVINTTERM(1,1)))):
> c4 := 5:
> C5:=BOUNDOP(EVOP(0.4,EVDIFFOP(0,1),EVINTOP(EVINTTERM(0,1)))):
> c5 := 6:
> C6:=BOUNDOP(EVOP(0.5,EVDIFFOP(0,1),EVINTOP(EVINTTERM(1,1)))):
> c6 := 7:
> C7:=BOUNDOP(EVOP(0.6,EVDIFFOP(1),EVINTOP(EVINTTERM(0,1)))):
> c7 := 9:
> C8:=BOUNDOP(EVOP(0.8,EVDIFFOP(1),EVINTOP(EVINTTERM(1,1)))):
> c8 := 13:
> C9:=BOUNDOP(EVOP(0.9,EVDIFFOP(0,1),EVINTOP(EVINTTERM(0,1)))):
> c9 := 15:
> C10:=BOUNDOP(EVOP(1.0,EVDIFFOP(1),EVINTOP(EVINTTERM(0,1)))):
> c10 := 16:
```

The following matrix shows the given Stieltjes conditions in integro-differential operators.

```
> C:=Matrix([[C1],[C2],[C3],[C4],[C5],[C6],[C7],[C8],[C9],[C10]]);
```

$$\begin{bmatrix} E[0] \\ E[.2] \\ E[.3].D \\ E[.3]+E[.3].A \\ E[.4].D \\ E[.5].D+E[.5].A \\ E[.6] \\ E[.8]+E[.8].A \\ E[.9].A \\ E[1.0] \end{bmatrix}$$

The following matrix gives the associated values of the given Stieltjes conditions.

```
> CM:=Matrix([[c1],[c2],[c3],[c4],[c5],[c6],[c7],[c8],[c9],[c10]]);
```

$$\begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 9 \\ 13 \\ 15 \\ 16 \end{bmatrix}$$

```
> PolynomialInterpolation(C, CM);
```

$$-1320.17293470549339\,x+642926.546278583002\,x^{8}-128730.827475196988\,x^{9}$$
$$+0.99999999996430911+21266.0762455070180\,x^{2}-141911.289549959474\,x^{3}$$
$$+522774.578675110242\,x^{4}-1167431.34558086051\,x^{5}+1621275.90023275116\,x^{6}$$
$$-1368834.46589123225\,x^{7}$$

Verification of the conditions

```
> f(x):= 1320.17293470549339 x + 642926.546278583002 x^8
     - 128730.827475196988 x^9 + 0.99999999996430911
```

```
+ 21266.0762455070180 x^2 – 141911.289549959474 x^3
+ 522774.578675110242 x^4 – 1167431.34558086051 x^5
+ 1621275.90023275116 x^6 – 1368834.46589123225 x^7;
```

$$x \to -1320.17293470549339\,x + 642926.546278583002\,x^8 -128730.827475196988\,x^9 + 0.99999999996430911 +21266.0762455070180\,x^2 -141911.289549959474\,x^3 +522774.578675110242\,x^4 -1167431.34558086051\,x^5 +1621275.90023275116\,x^6 -1368834.46589123225\,x^7$$

```
> f(0); f(0.2); f'(0.3); f(1.0);
```

$$1.00000000$$
$$3.00000044$$
$$4.00000000$$
$$15.9999999$$

Example 2: Recall an example of [5, Example 4] using the implementation. The authors in [5] computed the polynomial interpolation using manual calculations. Consider the conditions $\Theta = \{E_0, E_{\frac{1}{2}}, E_1 D, E_{\frac{4}{3}} A, E_2 A\}$ with $\Omega = \{1, 2, 0, 3, -1\}$.

```
> C1:=BOUNDOP(EVOP(0, VDIFFOP(1),EVINTOP(EVINTTERM(0,1))));
> C2:=BOUNDOP(EVOP(1/2,EVDIFFOP(1),EVINTOP(EVINTTERM(0,1))));
> C3:=BOUNDOP(EVOP(1,EVDIFFOP(0,1),EVINTOP(EVINTTERM(0,1))));
> C4:=BOUNDOP(EVOP(4/3,EVDIFFOP(0),EVINTOP(EVINTTERM(1,1))));
> C5:=BOUNDOP(EVOP(2,EVDIFFOP(0),EVINTOP(EVINTTERM(1,1))));
```

$$E[0]$$
$$E[1/2]$$
$$E[1].D$$
$$E[4/3].A$$
$$E[2].A$$

```
> C:=Matrix([[C1],[C2],[C3],[C4],[C5]]);
```

$$\begin{bmatrix} E[0] \\ E[1/2] \\ E[1].D \\ E[4/3].A \\ E[2].A \end{bmatrix}$$

```
> CM:=Matrix([[1],[2],[0],[3],[-1]]);
```

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ -1 \end{bmatrix}$$

```
> PolynomialInterpolation(C,CM);
```

$$1+\frac{21}{34}x+\frac{2505}{1088}x^2+\frac{1373}{544}x^3-\frac{435}{136}x^4.$$

```
> CM1:=Matrix([[c1],[c2],[c3],[c4],[c5]]):
> PolynomialInterpolation(C,CM1);
```

$$c1-\frac{39}{17}c1x+\frac{9}{272}c1x^2+\frac{181}{136}c1x^3-\frac{15}{34}c1x^4+\frac{384}{17}c2x$$
$$-\frac{1104}{17}c2x^2+\frac{928}{17}c2x^3-\frac{240}{17}c2x^4+\frac{62}{17}c3x-\frac{849}{68}c3x^2$$
$$+\frac{413}{34}c3x^3-\frac{60}{17}c3x^4-\frac{243}{17}c4x+\frac{48843}{1088}c4x^2-\frac{20169}{544}c4x^3$$
$$+\frac{1215}{136}c4x^4-\frac{21}{34}c5x+\frac{687}{272}c5x^2-\frac{441}{136}c5x^3+\frac{45}{34}c5x^4$$

After simplification of the above polynomial, we obtain

$$\tilde{f}_5(x) = \left(1 - \frac{39}{17}x + \frac{9}{272}x^2 + \frac{181}{136}x^3 - \frac{15}{34}x^4\right)c_0$$
$$+\left(\frac{384}{17}x - \frac{1104}{17}x^2 + \frac{928}{17}x^3 - \frac{240}{17}x^4\right)c_1$$
$$+\left(\frac{62}{17}x - \frac{849}{68}x^2 + \frac{413}{34}x^3 - \frac{60}{17}x^4\right)c_2$$
$$+\left(-\frac{243}{17}x + \frac{48843}{1088}x^2 - \frac{20169}{544}x^3 + \frac{1215}{136}x^4\right)c_3$$
$$+\left(-\frac{21}{34}x + \frac{687}{272}x^2 - \frac{441}{136}x^3 + \frac{45}{34}x^4\right)c_4.$$

```
> evlmat := EvaluationMat(C);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ 0 & 1 & 2 & 3 & 4 \\ \frac{4}{3} & \frac{8}{9} & \frac{64}{81} & \frac{64}{81} & \frac{1024}{1215} \\ 2 & 2 & \frac{8}{3} & 4 & \frac{32}{5} \end{bmatrix}$$

3. Conclusion.

In this paper, we presented a Maple implementation of a symbolic algorithm for polynomial interpolation with Stieltjes conditions at arbitrary points. Using the proposed Maple package, one can construct a polynomial interpolation with Stieltjes conditions. Sample computations are presented to illustrate the Maple package.

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