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MULTI-SUBLINEAR OPERATORS GENERATED BY MULTILINEAR FRACTIONAL INTEGRAL OPERATORS ON PRODUCT GENERALIZED MORREY SPACES

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Abstract. In this paper, we established the boundedness for a large class of multisublinear operators $T_{\alpha,m}$, $\alpha \in (0, mn)$ generated by multilinear fractional integral operator on product generalized Morrey spaces $M_{p_1,\varphi_1}(R^n)\times \ldots \times M_{p_m,\varphi_m}(R^n)$. We find the sufficient conditions on $(\varphi_1,\ldots,\varphi_m,\varphi)$ a which ensures the boundedness of the operators $T_{\alpha,m}$ from $M_{p_1,\varphi_1}(R^n)\times \ldots \times M_{p_m,\varphi_m}(R^n)$ to $M_{q,\varphi}(R^n)$ for $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n$. The multisublinear operators under consideration contain integral operators of harmonic analysis such as multi-sublinear fractional maximal operators $M_{\alpha,m}$, multilinear fractional integral operators $I_{\alpha,m}$, etc.

Keywords: multi-sublinear fractional maximal operator ; multilinear fractional integral operator ; product generalized Morrey space.

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1. Introduction.

Multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The initial work on the class of multilinear Calderón-Zygmund operators was done by Coifman and Meyer in [2] and was later systematically studied by Grafakos and Torres in [5,6].

The classical Morrey spaces, introduced by Morrey [18] in 1938, have been studied intensively by various authors and together with Lebesgue spaces play an important role in the theory of partial differential equations. Although such spaces allow to describe local properties of functions better than Lebesgue spaces, they have some unpleasant issues. It is well known that Morrey spaces are non separable and that the usual classes of nice functions are not dense in such spaces. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [8,16,19] introduced generalized Morrey spaces $M_{p,\varphi}(R^n)$

(see, also [9,10,14,20]). In [10] is defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm

$$\|f\|_{M_{p,\varphi}} \equiv \sup_{x\in R^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-1/p} \|f\|_{L_p(B(x,r))},$$

where the function φ is a positive measurable function on $R^n \times (0,\infty)$. Here and everywhere in the sequel $B(x,r)$ is the ball in R^n of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure , where v_n is the volume of the unit ball in R^n.

For $x \in R^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^C B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. We denote by $\vec{f}$ the m-tuple $(f_1, f_2, ..., f_m)$, $\vec{y} = (y_1, .., y_n)$ and $d\vec{y} = dy_1 ... dy_n$.

Let $\vec{f} \in L_{p_1}^{loc}(R^n) \times ... \times L_{p_m}^{loc}(R^n)$. The multi-sublinear fractional maximal operator $M_{\alpha,m}$ is defined by

$$M_{\alpha,m}(\vec{f})(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha}{n}} \prod_{j=1}^{m} \frac{1}{|B(x,r)|} \int_{B(x,r)} f_i(y_i) dy_i, \quad 0 \le \alpha < mn.$$

In [15] Kenig and Stein studied the following multilinear fractional integral,

$$I_{\alpha,m}(\vec{f})(x) = \int_{(R^n)^m} \frac{f_1(y_1) ... f_m(y_m)}{|(x - y_1, ..., x - y_m)|^{mn-\alpha}} dy_1 dy_2 ... dy_m,$$

and showed that $I_{\alpha,m}$ is bounded from product $L_{p_1}(R^n) \times ... \times L_{p_m}(R^n)$ to $L_q(R^n)$ with $1/q = 1/p_1 + ... + 1/p_m - \beta/n > 0$ for each $p_i > 1$ $(i = 1,..,m)$. If some $p_i = 1$, then $I_{\alpha,m}$ is bounded from $L_{p_1}(R^n) \times ... \times L_{p_m}(R^n)$ to $WL_q(R^n)$, where $WL_q(R^n)$ denotes the weak L_p-space of measurable functions on R^n . Obviously, the multilinear fractional integral operator $I_{\alpha,m}$ is a natural generalization of the classical fractional integral operator $I_\alpha \equiv I_{\alpha,1}$.

It is well known that multi-sublinear fractional maximal operator and multilinear fractional integral operator play an important role in harmonic analysis (see, also [1,4,15,17]).

Suppose that $T_{\alpha,m}$, $\alpha \in (0, mn)$ represents a multilinear or a multi-sublinear operator, which satisfies that for any $\vec{f} \in L_1(R^n) \times \ldots \times L_1(R^n)$ with compact support and $x \notin \bigcap_{j=1}^{m} \sup pf_j$

$$\left| T_{\alpha,m}\left(\vec{f}\right)(x) \right| \le c_0 \int_{(R^n)^m} \frac{|f_1(y_1)\ldots f_m(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}} dy_1 dy_2 \ldots dy_m \quad (1)$$

for some $\alpha \in (0, mn)$, where c_1 is independent of f and x.

The condition (1) is satisfied by many interesting operators in harmonic analysis, such as the multi-sublinear fractional maximal operator, multilinear fractional integral operator, and so on (see [4,12,15] for details).

In this work, we prove the boundedness of the multi-sublinear operator $T_{\alpha,m}$, $\alpha \in (0, mn)$ satisfies the condition (1) generated by multilinear fractional integral operator from $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $M_{q,\varphi}$, if $1 < p_1,\ldots,p_m < \infty$ and $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n$, and from the space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to the weak space $WM_{q,\varphi}$, if $1 \le p_1,\ldots,p_m < \infty$, $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n$ and at least one p_i equals one (Theorem 2.3). Finally, as applications we apply this result to several particular operators such as the multi-sublinear fractional maximal operator and multilinear fractional integral operator.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. .

2. Main Results

In this section, we will prove the boundedness of multi-sublinear operators $T_{\alpha,m}$, $\alpha \in (0, mn)$ generated by multilinear fractional integral operator on product generalized Morrey spaces $M_{p_1,\varphi_1}(R^n) \times \ldots \times M_{p_m,\varphi_m}(R^n)$.

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.1. *Let* $\varphi(x,r)$ *be a positive measurable function on* $R^n \times (0,\infty)$ *and* $1 \le p < \infty$. *We denote by* $M_{p,\varphi} \equiv M_{p,\varphi}(R^n)$ *the generalized Morrey space, the space of all functions* $f \in L_p^{loc}(R^n)$ *with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in R^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(R^n)$ *we denote the weak generalized Morrey space of all functions* $f \in WL_p^{loc}(R^n)$ *for which*

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in R^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty$$

Lemma 2.1. [3] *Let* $\varphi(x,r)$ *be a positive measurable function on* $R^n \times (0,\infty)$.
(i) *If*

$$\sup_{t<r<\infty} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} = \infty \quad \text{for some } t>0 \text{ and for all } x \in R^n, \tag{2}$$

then $M_{p,\varphi}(R^n) = \Theta$.
(ii) *If*

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \text{for some } \tau>0 \text{ and for all } x \in R^n, \tag{3}$$

then $M_{p,\varphi}(R^n) = \Theta$.

Remark 2.1. *We denote by* Ω_p *the sets of all positive measurable functions* φ *on* $R^n \times (0,\infty)$ *such that for all* $t>0$,

$$\sup_{x\in R^n} \left\| \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} \right\|_{L_\infty(t,\infty)} < \infty, \qquad \text{and} \qquad \sup_{x\in R^n} \left\| \varphi(x,r)^{-1} \right\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.1, we always assume that $\varphi \in \Omega_p$.

We will use the following statements on the boundedness of the weighted Hardy operator

$$H_w g(r) := \int_r^\infty g(t) w(t) dt \quad , \quad 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.

The following theorem was proved in [11] (see also [13]).

Theorem 2.1. [11] *Let* v_1, v_2 *and* w *be positive almost everywhere and measurable functions on* $(0,\infty)$. *The inequality*

$$\operatorname*{ess\,sup}_{r>0} v_2(r) H_w g(r) \le C \operatorname*{ess\,sup}_{r>0} v_1(r) g(r) \tag{4}$$

holds for some $C > 0$ *for all non-negative and non-decreasing g on* $(0,\infty)$ *if and only if*

$$B := \sup_{r>0} v_2(r) \int_r^\infty \frac{w(t)dt}{\sup_{t<s<\infty} v_1(s)} < \infty \tag{5}$$

Moreover, the value $C = B$ *is the best constant for* (4).

Remark 2.2. *In* (4)-(5) *it is assumed that* $0 \cdot \infty = 0$.

In the following lemma we get Guliyev local estimate (see, for example, [8,9,10] in the case $m=1$ and [12] in the case $m>1$) for the operator $T_{\alpha,m}$.

Theorem 2.2. *Let* $1 \le p_1, \ldots, p_m < \infty$ *and* $0 < \alpha < mn$ *with* $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n$ *and* $\alpha = \sum_{i=1}^m \alpha_i$ *where each* α_i *satisfies* $0 < \alpha_i < \frac{n}{p_i}$. *Let also* $T_{\alpha,m}$ *be a multi-sublinear operator which satisfies the condition* (1) *and bounded from* $L_{p_1}(R^n) \times \ldots \times L_{p_m}(R^n)$ *to* $L_q(R^n)$ *for* $p_i > 1$, $i = 1,\ldots,m$, *and bounded from* $L_{p_1}(R^n) \times \ldots \times L_{p_m}(R^n)$ *to* $WL_q(R^n)$ *for* $p_i \ge 1$, $i = 1,\ldots,m$. *Then for* $p_i > 1$, $i = 1,\ldots,m$ *the inequality*

$$\left\| T_{\alpha,m}(\vec{f})(x) \right\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \prod_{i=1}^m \int_r^\infty t^{\alpha_i - \frac{n}{p_i} - 1} \| f_i \|_{L_{p_i}(B(x_0,t))} dt, \tag{6}$$

holds for any ball $B(x_0, r)$ *and for all* $\vec{f} \in L_{p_1}^{loc}(R^n) \times \ldots \times L_{p_m}^{loc}(R^n)$ *Moreover, if at least one* p_i *equals one the inequality*

$$\left\| T_{\alpha,m}(\vec{f})(x) \right\|_{WL_q(B(x,r))} \lesssim r^{\frac{n}{q}} \prod_{i=1}^m \int_r^\infty t^{\alpha_i - \frac{n}{p_i} - 1} \| f_i \|_{L_{p_i}(B(x,t))} dt, \tag{7}$$

holds for any ball $B(x_0, r)$ *and for all* $\vec{f} \in L_{p_1}^{loc}(R^n) \times \ldots \times L_{p_m}^{loc}(R^n)$.

Proof. Let $1 \le p_1,..., p_m < \infty$ and $1/q = 1/p_1 + ... + 1/p_m - \alpha/n.$ For arbitrary $x_0 \in R^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0, 2r)$. We represent $\vec{f} = (f_1,..., f_m)$ as

$$f_j = f_j^0 + f_j^\infty, \qquad f_j^0 = f_j \chi_{2B}, \qquad f_j^\infty = f_j \chi_{C_{(2B)}}, \; j = 1,\dots,m. \tag{8}$$

Then we split $T_{\alpha,m}(\vec{f})(x)$ as following ,

$$\left|T_{\alpha,m}(\vec{f})(x)\right| \le c_0 \left|T_{\alpha,m}(f_1^0,..., f_m^0)(x)\right| + \left|\sum_{\beta_1,...,\beta_m}{}' T_{\alpha,m}(f_1^{\beta_1},..., f_m^{\beta_m})(x)\right|,$$

where $\beta_1,...,\beta_m \in \{0, \infty\}$ and each term of $\sum{}'$ contains at least $\beta_i \ne 0$. Then,

$$\left\|T_{\alpha,m}(\vec{f})\right\|_{L_p(B(x,r))} \le \left\|T_{\alpha,m}(f^0)\right\|_{L_p(B(x,r))} + \left\|\sum_{\beta_1,...,\beta_m}{}' T_{\alpha,m}(f_1^{\beta_1},..., f_m^{\beta_m})\right\|_{L_p(B(x,r))}$$

$$\le I + II.$$

For I, by the boundedness of $T_{\alpha,m}$ on product L_p spaces, we have,

$$\left\|T_{\alpha,m}(\vec{f}^{\,0})\right\|_{L_p(B(x,r))} \le \left\|T_{\alpha,m}(\vec{f}^{\,0})\right\|_{L_p(R^n)} \lesssim \prod_{i=1}^m \left\|f_i^0\right\|_{L_{p_i}(R^n)} \lesssim \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,2r))}.$$

Taking into account that

$$\|f_i\|_{L_{p_i}(B(x,2r))} \lesssim r^{\frac{n}{p_i} - \alpha_i} \int_{2r}^\infty t^{\alpha_i - \frac{n}{p_i} - 1} \|f_i\|_{L_{p_i}(B(x,t))} dt, \quad i = 1,......, m$$

we get

$$\left\|T_{\alpha,m}(\vec{f}^{\,0})\right\|_{L_q(B(x,r))} \lesssim r^{\frac{n}{q}} \prod_{i=1}^m \int_{2r}^\infty t^{\alpha_i - \frac{n}{p_i} - 1} \|f_i\|_{L_{p_i}(B(x,t))} dt. \tag{9}$$

For II, first we consider the case $\beta_1 = \beta_2 = ... = \beta_m = \infty$.

When $|x - y_i| \le r$, $|z - y_i| \ge 2r$, we have $\frac{1}{2}|z - y_i| \le |x - y_i| \le \frac{3}{2}|z - y_i|$, and therefore

$$\left\|T_{\alpha,m}\left(\vec{f}^{\infty}\right)\right\|_{L_q(B(x,r))} \leq \left\|\int_{{}^C(B(x,2r))^m} \frac{f_1(y_1)\ldots f_m(y_m)}{|(z-y_1,\ldots,z-y_m)|^{mn-\alpha}}\,d\vec{y}\right\|_{L_q(B(x,r))}$$

$$\lesssim \int_{{}^C(B(x,2r))^m} \frac{|f_1(y_1)\ldots f_m(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}}\,d\vec{y}\,\|\chi_{B(x,r)}\|_{L_q(R^n)}$$

$$\lesssim r^{\frac{n}{q}}\prod_{i=1}^{m}\int_{{}^C B(x,2r)}|x-y_i|^{\alpha_i-n}\,|f_i(y_i)|dy_i.$$

From inequality (5.9) in $[10]$, for any $p \geq 1$ we have

$$\int_{{}^C B(x,2r)}|x-y|^{\beta-n}\,|f(y)|dy \lesssim \int_{2r}^{\infty} t^{\beta-\frac{n}{p}-1}\,\|f\|_{L_p(B(x,t))}dt, \tag{10}$$

so we have

$$\left\|T_{\alpha,m}\left(\vec{f}^{\infty}\right)\right\|_{L_q(B(x,r))} \lesssim \prod_{i=1}^{m} r^{\frac{n}{q}}\int_{r}^{\infty} t^{\alpha_i-\frac{n}{p_i}-1}\,\|f_i\|_{L_{p_i}(B(x,t))}dt.$$

Finally, for the case that $\beta_{j1}=\beta_{j2}=\ldots=\beta_{jl}=0$ for some $\{j1,\ldots,jl\}\subset\{1,..,m\}$ where $1\leq l<m$, we only consider the case only $\beta_1=0$ since the other ones follow in analogous way.

Choose a series positive numbers $q_1, q_{2,\ldots,} q_m$ which satisfy $1/p_i - 1/q_i = \alpha_i/n \quad (i=1,\ldots,m)$ such that $\sum_{i=1}^{m} 1/q_i = 1/q$, we have

$$\left\|T_{\alpha,m}\left(f_1^0, f_2^{\infty},\ldots, f_m^{\infty}\right)\right\|_{L_q(B(x,r))}$$

$$\lesssim \left\|T_{\alpha,m}\left(f_1^0\right)\right\|_{L_{q_1}(B(x,r))}\prod_{i=2}^{m}\left\|\int_{{}^C B(x,2r)}\frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha_i}}dy_i\right\|_{L_{q_i}(B(x,r))}$$

$$\lesssim \|f_1\|_{L_{p_1}(B(x,r))}\prod_{i=2}^{m}\left|\int_{{}^C B(x,2r)}\frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha}}dy_i\right|\|\chi_{B(x,r)}\|_{L_{q_i}(R^n)}$$

$$\lesssim r^{\frac{n}{p_i}-\alpha_i}\int_{2r}^{\infty} t^{\alpha_i-\frac{n}{p_i}-1}\|f_1\|_{L_{p_1}(B(x,t))}dt\prod_{i=2}^{m} r^{\frac{n}{q_i}}\int_{2r}^{\infty} t^{\alpha_i-\frac{n}{p_i}-1}\|f_i\|_{L_{p_i}(B(x,t))}dt$$

$$\leq r^{\frac{n}{q}}\prod_{i=1}^{m}\int_{r}^{\infty} t^{\alpha_i-\frac{n}{p_i}-1}\|f_i\|_{L_{p_i}(B(x,t))}dt.$$

For the proof of inequality (7) , pay attention to the fact that $I_{\beta,m}$ is bounded from $L_{p_1}(R^n)\times\ldots\times L_{p_m}(R^n)$ to $WL_q(R^n)$ for at least one p_i equals one if $\sum_{i=1}^{m} 1/p_i - 1/q_i = \alpha/n$, then by a similar argument above can we easily prove inequality (7).

So far, the proof of Theorem 2.2 has been finished.

Now we give the boundedness of multilinear fractional integral on product generalized Morrey spaces.

Theorem 2.3. *Let* $1 \le p_1,\ldots,p_m < \infty$, $0<\alpha<mn$ *with* $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n$ *and* $\alpha = \sum_{i=1}^{m}\alpha_i$ *where each* α_i *satisfies* $0<\alpha_i<\frac{n}{p_i}$. *Let also* $(\varphi_1,\ldots,\varphi_m,\varphi)\in\Omega_{p_1}\times\ldots\times\Omega_{p_1}\times\Omega_q$ *satisfies the condition*

$$\prod_{i=1}^{m}\int_r^\infty \frac{\operatorname{ess\,inf}\limits_{t<s<\infty}\varphi_i(x,s)s^{\frac{n}{p_i}}}{t^{\frac{n}{q_i}+1}}dt \lesssim \varphi(x,r). \tag{11}$$

Let also $T_{\alpha,m}$ *be a multi-sublinear operator which satisfies the condition* (1) *and bounded from* $L_{p_1}(R^n)\times\ldots\times L_{p_m}(R^n)$ *to* $L_q(R^n)$ *for* $p_i>1$, $i=1,\ldots,m$, *and bounded from* $L_{p_1}(R^n)\times\ldots\times L_{p_m}(R^n)$ *to* $WL_q(R^n)$ *for* $p_i\ge 1$, $i=1,..,m$. *Then the operator* $T_{\alpha,m}$ *is bounded from product space* $M_{p_1,\varphi_1}(R^n)\times\ldots\times M_{p_m,\varphi_m}(R^n)$ *to* $M_{q,\varphi}(R^n)$ *for* $p_i>1$, $i=1,\ldots,m$ *and from product space* $M_{p_1,\varphi_1}(R^n)\times\ldots\times M_{p_m,\varphi_m}(R^n)$ *to* $WM_{q,\varphi}(R^n)$ *for at least one* p_i *equals one.*

Proof. Let $1<p_1,..,p_m<\infty$ and $\vec{f}\in M_{p_1,\varphi_1}(R^n)\times\ldots\times M_{p_m,\varphi_m}(R^n)$. By Theorems 2.1 and 2.2 we have

$$\left\|T_{\alpha,m}\left(\vec{f}\right)\right\|_{M_{q,\varphi}} \lesssim \sup_{x\in R^n,r>0}\varphi(x,r)^{-1}\prod_{i=1}^{m}\int_r^\infty t^{-\frac{n}{q_i}-1}\|f_i\|_{L_{p_i}(B(x,t))}dt$$

$$\lesssim \prod_{i=1}^{m}\sup_{x\in R^n,r>0}\varphi_i(x,r)^{-1}r^{\frac{q_i}{p_i}}\|f_i\|_{L_{p_i}(B(x,r))} = \prod_{i=1}^{m}\|f_i\|_{M_{p_i,\varphi_i}}.$$

When $p_i=1$ $(i=1,..,m)$, the proof is similar and we omit the details here.

Corollary 2.1. [12] *Let* $1 \le p_1,..., p_m < \infty$, $0 < \alpha < mn$ *with* $1/q = 1/p_1 + ... + 1/p_m - \alpha/n$ *and* $\alpha = \sum_{i=1}^{m} \alpha_i$ *where each* α_i *satisfies* $0 < \alpha_i < \frac{n}{p_i}$. *Let also* $(\varphi_1,..,\varphi_m,\varphi) \in \Omega_{p_1} \times ... \times \Omega_{p_1} \times \Omega_q$ *satisfies the condition* (11).

Then the operators $I_{\alpha,m}$ *and* $M_{\alpha,m}$ *are bounded from product space* $M_{p_1,\varphi_1}(R^n) \times ... \times M_{p_m,\varphi_m}(R^n)$ *to* $M_{q,\varphi}(R^n)$ *for* $p_i > 1$, $i = 1,..., m$ *and from product space* $M_{p_1,\varphi_1}(R^n) \times ... \times M_{p_m,\varphi_m}(R^n)$ *to* $WM_{q,\varphi}(R^n)$ *for at least one* p_i *equals one.*

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REFERENCES

1. Coifman R., Grafakos L. Hardy spaces estimates for multilinear operators I, Rev. Math. Iberoamericana, V.8, 1992, pp.45-68.
2. Coifman R.R., Meyer Y. On commutators of singular integrals and bilinear singular integrals, Trans. Amer. Math. Soc., V.212, 1975, pp.315-331.
3. Eroglu A., Guliyev V.S., Azizov C.V. Characterizations for the fractional integral operators in generalized Morrey spaces on Carnot groups, Math. Notes., V.102, N.5-6, 2017, pp.722-734.
4. Grafakos L. On multilinear fractional integrals, Studia Math., V.102, 1992, pp.49-56.
5. Grafakos L., Torres R.H. Multilinear Caldern-Zygmund theory, Advances in Mathematics, V.165, N.1, 2002, pp.124-164.
6. Grafakos L., Torres R.H. On multilinear singular integrals of Calderón-Zygmund type, Publicacions Matem`atiques, V.46, 2002, pp.57-91.
7. Guliyev V.S. Integral operators on function spaces on the homogeneous groups and on domains in R^n. Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 1994, 329 p. (in Russian)
8. Guliyev V.S. Integral operators on function spaces on the homogeneous groups and on domains in R^n. Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 1994, 329 p. (in Russian)

9. Guliyev V.S. Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups, Some Applications, Casioglu, Baku, 1999, 332 p. (in Russian)
10. Guliyev V.S. Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces, J. Inequal. Appl., 2009, Art. ID 503948, 20 p.
11. Guliyev V.S. Generalized weighted Morrey spaces and higher order commutators of sublinear operators, Eurasian Math. J., V.3, N.3, 2012, pp.33-61.
12. Guliyev V.S., Ismayilova A.F. Multi-sublinear fractional maximal operator and multilinear fractional integral operators on generalized Morrey spaces, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., V.40, N.2, 2014 pp.22-33.
13. Guliyev V.S., Omarova M.N., Ragusa M.A., Scapellato A. Commutators and generalized local Morrey spaces, J. Math. Anal. Appl., V.457, N.2, 2018, pp.1388-1402.
14. Guliyev V.S., Guliyev R.V., Omarova, M.N. Riesz transforms associated with Schrödinger operator on vanishing generalized Morrey spaces, Appl. Comput. Math., V.17, N.1, 2018, pp.56-71.
15. Kenig C.E., Stein E.M. Multilinear estimates and fractional integration, Math. Res. Lett., V.6, 1999, pp.1-15.
16. Mizuhara T. Boundedness of some classical operators on generalized Morrey spaces, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer Verlag, Tokyo, 1991, pp.183-189
17. Moen K. Weighted inequalities for multilinear fractional integral operators, Collect. Math, V.60, 2009, pp.213-238.
18. Morrey C.B., On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc, V.43, 1938, pp.126-166.
19. Nakai E. Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces, Math. Nachr, V.166, 1994, pp.95-103.
20. Sawano Y. A thought on generalized Morrey spaces, J. Indonesian Math. Soc., V.25, N.3, 2019, pp.210-281.